DAWSON COLLEGE

MATHEMATICS DEPARTMENT

Linear Algebra

201-NYC-05 5
Fall 2015
Final Exam
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Time Limit: 3 hours

Name: Solutions

ID#:

- This test contains 14 pages (including this cover page) and 20 problems. Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- You are only permitted to use the **Sharp EL-531XG** or **Sharp EL-531X** calculator.
- This examination booklet must be returned intact.
- Good luck!

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	11	
10	5	
11	3	
12	3	
13	8	
14	3	
15	3	
16	4	
17	4	
18	4	
19	4	
20	8	
Total:	100	

1. Find the general solution set of the system given by $\begin{cases} x+2y-z+w=2 \\ x-y+z+w=0 \\ 2x+y-w=1 \\ 4x+2y+w=3 \end{cases}$ $\boxed{5}$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 1 & 3 \end{array}\right) \xrightarrow[R_4-4R_1]{\substack{R_2-R_1 \\ R_3-2R_1}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -3 & 2 & 0 & -2 \\ 0 & -3 & 2 & -3 & -3 \\ 0 & -6 & 4 & -3 & -5 \end{array}\right)$$

$$\xrightarrow[R_4-2R_2]{R_3-R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -3 & 2 & 0 & -2 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -3 & -1 \end{array}\right) \xrightarrow[-1/3 R_2]{R_4-R_3} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -2/3 & 0 & 2/3 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-1/3 R_3]{R_1-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1/3 & 1 & 2/3 \\ 0 & 1 & -2/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1 & -2/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \begin{cases} x = 1/3 - 1/3\, t \\ y = 2/3 + 2/3\, t \\ z = t \\ w = 1/3 \end{cases}$$

or $\underline{X} = (1/3, 2/3, 0, 1/3) + t(-1/3, 2/3, 1, 0)$.

2. Given $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & -3 \end{pmatrix}$. Find $\text{tr}(X)$ if X satisfies the equation $\boxed{5}$

$$(X + BC)^{-1} = A.$$

$X + BC = A^{-1}$

$X = A^{-1} - BC$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}^{-1} - \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & -3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & -3 & 9 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 3 & -8 \end{pmatrix}$ $\Rightarrow tr(X) = -1 + 1 - 8 = -8$

3. Prove: Any matrix A can be expressed as $A = ER$ where E is an invertible matrix and R is a matrix in reduced row-echelon form. $\boxed{5}$

If A reduces to R in RREF then we can write

$E_n \cdots E_1 A = R$ where $E_1, \ldots, E_n$ are the elementary matrices that reduces A to R.

$\Rightarrow A = (E_n \cdots E_1)^{-1} R$ since each E_i is invertible.

and $A = ER$

where $E = E_1^{-1} \cdots E_n^{-1}$.

4. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ using the adjoint. [5]

$C_{11} = 0 - 2 = -2$ $\quad C_{12} = -(-1-6) = 7$ $\quad C_{13} = 1 - 0 = 1$

$C_{21} = -(-2-2) = 4$ $\quad C_{22} = -1 - 6 = -7$ $\quad C_{23} = -(1-6) = 5$

$C_{31} = 4 - 0 = 4$ $\quad C_{32} = \cancel{(\quad)} = -(2-2) = 0$ $\quad C_{33} = 0 - 2 = -2$.

$\det A = 2(7) + 0(-7) + 1(0) = 14$ (column 2).

$$\Rightarrow A^{-1} = \frac{1}{14} \begin{pmatrix} -2 & 7 & 1 \\ 4 & -7 & 5 \\ 4 & 0 & -2 \end{pmatrix}^T$$

$$\boxed{= \begin{pmatrix} -1/7 & 2/7 & 2/7 \\ 1/2 & -1/2 & 0 \\ 1/14 & 5/14 & -1/7 \end{pmatrix}}$$

5. Find all values of t so that $A = \begin{pmatrix} t+4 & 1 & -1 \\ 8 & t-3 & 12 \\ -4 & 4 & t+5 \end{pmatrix}$ is not invertible. [5]

$$\det A = \begin{vmatrix} t+4 & 1 & -1 \\ 8 & t-3 & 12 \\ -4 & 4 & t+5 \end{vmatrix} \begin{matrix} C_1 + C_2 \\ = \\ C_3 + C_2 \end{matrix} \begin{vmatrix} t+5 & 1 & 0 \\ t+5 & t-3 & t+9 \\ 0 & 4 & t+9 \end{vmatrix}$$

$$= (t+5)\big((t-3)(t+9) - 4(t+9)\big) - (t+5)(t+9)$$

$$= (t+5)(t+9)(\cancel{t}-3-4) - (t+5)(t+9)$$

$$= (t+5)(t+9)(t-7-1)$$

$$= (t+5)(t+9)(t-8)$$

$$\Rightarrow t = -5, -9, \text{ or } 8$$

6. Given the system of linear equations $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$, solve for z in terms of t using Cramer's Rule. (Simplify your answer.) [5]

$$\det A = (e^t \cos t)(e^t \cos t) + (e^t \sin t)(e^t \sin t)$$

$$= e^{2t}(\cos^2 t + \sin^2 t) = e^{2t}$$

$$\det A_z = \begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix} = t^2 \cdot e^t \cos t + t e^t \sin t$$

$$= t e^t (t \cos t + \sin t).$$

$$\therefore\ z = \frac{t e^t (t \cos t + \sin t)}{e^{2t}} = \boxed{\frac{t(t\cos t + \sin t)}{e^t}}$$

7. Let A be a 4×4 matrix such that $\det(A) = -3$. Find $\boxed{5}$

$$\det\left((2A)^{-1} + \operatorname{adj}(A)\right)$$

$$= \det\left(\tfrac{1}{2}A^{-1} + (\det A)A^{-1}\right) = \det\left(\tfrac{1}{2}A^{-1} - 3A^{-1}\right)$$

$$= \det\left(-\tfrac{5}{2}A^{-1}\right) = \left(-5/2\right)^4 \cdot \det A^{-1}$$

$$= \frac{625}{16} \cdot \frac{1}{-3} = -\frac{625}{48}$$

8. If $A = \begin{pmatrix} k & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, show that $\det(AA^T) \neq \det(A^TA)$ for every $k \in \mathbb{R}$. $\boxed{5}$

$$|AA^T| = \left|\begin{pmatrix} k & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}\right| = \begin{vmatrix} k^2+10 & 3 \\ 3 & 1 \end{vmatrix} = k^2+10-9 = k^2+1$$

$$|A^TA| = \left|\begin{pmatrix} k & 0 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} k^2 & 3k & k \\ 3k & 10 & 3 \\ k & 3 & 1 \end{pmatrix}\right|$$

$$= k^2(10-9) - 3k(3k-3k) + k(9k-10k)$$

$$= k^2 - k^2 = 0$$

$\Rightarrow$ need $k^2+1=0$ or $k^2=-1$ $\rightarrow\leftarrow$

A= B=

9. Let $A = (3, -1, 2)$ be a point, and let L be the line through the points $(2, 1, 4)$, and $(-3, -1, 7)$.

(a) Find the equation of the plane passing through A and orthogonal to L. [4]

$\underline{N} = \vec{AB} = \langle -5, -2, 3 \rangle$, $A = (3, -1, 2)$

$$-5x - 2y + 3z = -5(3) - 2(-1) + 3(2)$$
$$= -15 + 2 + 6 = -7$$

$$\therefore \boxed{-5x - 2y + 3z = -7}$$

(b) Find the point of intersection of L with the plane found in part (a). [4]

$\underline{V} = \langle -5, -2, 3 \rangle \Rightarrow L: \langle 2, 1, 4 \rangle + t \langle -5, -2, 3 \rangle$

$$\Rightarrow \begin{cases} x = 2 - 5t \\ y = 1 - 2t \\ z = 4 + 3t \end{cases}$$

$$-5(2-5t) - 2(1-2t) + 3(4+3t) = -7$$
$$-10 + 25t - 2 + 4t + 12 + 9t = -7$$
$$38t = -7$$
$$t = -7/38$$

$$\Rightarrow \begin{matrix} x = 2 + 35/38 = 111/38 \\ y = 1 + 14/38 = 52/38 = 26/19 \\ z = 4 - 21/38 = 131/38 \end{matrix}$$

$\therefore POI = (111/38, 26/19, 131/38).$

(c) Use this point to find the distance from A to L. [3]

$$d = \sqrt{(3 - 111/38)^2 + (-1 - 26/19)^2 + (2 - 131/38)^2}$$
$$= \sqrt{\left(\frac{3}{38}\right)^2 + \left(\frac{-90}{38}\right)^2 + \left(\frac{-55}{38}\right)^2}$$
$$= \frac{\sqrt{3^2 + (-90)^2 + (-55)^2}}{38} = \boxed{\frac{\sqrt{11134}}{38}}$$

10. Find the equation of the plane which contains the lines $\begin{cases} x = s \\ y = 1 - s \\ z = 3 + 4s \end{cases}$ and $\begin{cases} x = t \\ y = -1 + t \\ z = 5 + 2t \end{cases}$. $\boxed{5}$

$$\underline{N} = \langle 1, -1, 4 \rangle \times \langle 1, 1, 2 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 4 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \langle -2-4, 4-2, 1+1 \rangle = \langle -6, 2, 2 \rangle.$$

use $\underline{N} = \langle -3, 1, 1 \rangle$.

and $A = (0, 1, 3)$.

$$-3x + y + z = -3(0) + 1 + 3 = 4$$

$$-3x + y + z = 4$$

11. Prove: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ generate a parallelepiped, then its volume is equal to $\|\mathbf{u} \times \mathbf{v}\|^2$. $\boxed{3}$

$$V = |\underline{u} \cdot (\underline{v} \times (\underline{u} \times \underline{v}))|$$

$$= |\underline{u} \times \underline{v} \cdot (\underline{v} \times \underline{u})|$$

$$= |(\underline{u} \times \underline{v}) \cdot (\underline{u} \times \underline{v})|$$

$$= \|\underline{u} \times \underline{v}\|^2$$

12. Simplify: $\|\mathbf{u} \times \mathbf{v}\|^2 + (\mathbf{u} \cdot \mathbf{v})^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2$. $\boxed{3}$

$$= \|\underline{u}\|^2 \|\underline{v}\|^2 \cdot \sin^2\theta + \|\underline{u}\|^2 \cdot \|\underline{v}\|^2 \cos^2\theta - \|\underline{u}\|^2 \|\underline{v}\|^2$$

$$= \|\underline{u}\|^2 \cdot \|\underline{v}\|^2 \left(\sin^2\theta + \cos^2\theta - 1 \right)$$

$$= 0$$

13. Consider the lines $\begin{cases} x = 2+s \\ y = 1+s \\ z = -2+s \end{cases}$ and $\begin{cases} x = 1+2t \\ y = 2-t \\ z = 1-t \end{cases}$.

P Q

$v_1 = \langle 1, 1, 1 \rangle$

$v_2 = \langle 2, -1, -1 \rangle$

(a) Find the points in each line that are nearest to each other. [5]

$\vec{PQ} = \langle -1+2t-s, 1-t-s, 3-t-s \rangle$

$\begin{cases} \vec{PQ} \cdot v_1 = 0 \\ \vec{PQ} \cdot v_2 = 0 \end{cases} \Rightarrow \begin{cases} -1+2t-s+1-t-s+3-t-s = 0 \\ 2(-1+2t-s)-(1-t-s)-(3-t-s) = 0 \end{cases} \Rightarrow \begin{cases} -3s = -3 \\ 6t = 6 \end{cases}$

$\Rightarrow s = 1$ and $t = 1$.

$\therefore \quad P = (2+1, 1+1, -2+1) = (3, 2, -1)$

and $Q = (1+2, 2-1, 1-1) = (3, 1, 0)$

(b) Find the shortest distance between the lines. [3]

$d = \|\vec{PQ}\| = \sqrt{(3-3)^2 + (1-2)^2 + (0+1)^2}$

$= \sqrt{2}$

14. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that if $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{w}$, then $\mathbf{u} - \mathbf{w}$ is orthogonal to $\mathbf{v}$. $\boxed{3}$

we have that $\dfrac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|^2}\underline{v} = \dfrac{\underline{w} \cdot \underline{v}}{\|\underline{v}\|^2}\underline{v}$

$\Rightarrow \quad \underline{u} \cdot \underline{v} = \underline{w} \cdot \underline{v}$

$\Rightarrow \quad \underline{u} \cdot \underline{v} - \underline{w} \cdot \underline{v} = 0$

$\Rightarrow \quad (\underline{u} - \underline{w}) \cdot \underline{v} = 0$

and $\underline{u} - \underline{w}$ is orthogonal to $\underline{v}$.

15. Prove or give a counter-example: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. $\boxed{3}$

Let $\underline{u} = \underline{v} = \hat{i} = \langle 1, 0, 0 \rangle$ and $\underline{w} = \hat{j} = \langle 0, 1, 0 \rangle$.

then $\underline{u} \times (\underline{v} \times \underline{w}) = \hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$

and $(\underline{u} \times \underline{v}) \times \underline{w} = (\hat{i} \times \hat{i}) \times \hat{j} = \underline{0} \times \hat{j} = \underline{0}$

$\therefore \quad \underline{u} \times (\underline{v} \times \underline{w}) \neq (\underline{u} \times \underline{v}) \times \underline{w}$

in general.

16. Find a geometric description for span $\left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}, \begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$. $\boxed{4}$

Let $\underline{x} \in$ Span $(\cdots)$ with $\underline{x} = (x, y, z)$.

then we have $\left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & x \\ 2 & 4 & 3 & 1 & y \\ 0 & 0 & 1 & 1 & z \end{array}\right)$

$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & x \\ 0 & 0 & 1 & 1 & y-2x \\ 0 & 0 & 1 & 1 & z \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & x \\ 0 & 0 & 1 & 1 & y-2x \\ 0 & 0 & 0 & 0 & z-y+2x \end{array}\right).$

$\Rightarrow \underline{x}$ satisfies $2x - y + z = 0$

which is a plane through the origin perpendicular to $\langle 2, -1, 1 \rangle$ in $\mathbb{R}^3$.

17. Given non-zero orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, show that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set. $\boxed{4}$

Suppose $c_1 \underline{u} + c_2 \underline{v} = \underline{0}$

then $\underline{u} \cdot (c_1 \underline{u} + c_2 \underline{v}) = \underline{u} \cdot \underline{0}$

$c_1 \underline{u} \cdot \underline{u} + c_2 \underbrace{\underline{u} \cdot \underline{v}}_{0} = 0$

$\Rightarrow c_1 \cdot \|\underline{u}\|^2 = 0 \Rightarrow c_1 = 0$ since $\underline{u}$ is non-zero

Similarly, $c_2 = 0$.

$\Rightarrow$ linearly independent.

18. Is this set of matrices $\left\{\begin{pmatrix}1&2\\3&0\end{pmatrix}, \begin{pmatrix}2&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&2\end{pmatrix}, \begin{pmatrix}3&2\\2&-1\end{pmatrix}\right\}$ linearly independent? If it is dependent, exhibit one of the matrices as a linear combination of the others. $\boxed{4}$

by inspection $\begin{pmatrix}3&2\\2&-1\end{pmatrix} = \begin{pmatrix}1&2\\3&0\end{pmatrix} + \begin{pmatrix}2&1\\0&1\end{pmatrix} - \begin{pmatrix}0&1\\1&2\end{pmatrix}$

$\Rightarrow$ dependent.

19. Let $A = \begin{pmatrix}1&1&2&0&1\\-2&-1&1&-3&0\\3&2&1&3&1\\4&2&-2&6&0\end{pmatrix}$. Find a basis for, and the dimension of, the solution space of $A\mathbf{x} = \mathbf{0}$. $\boxed{4}$

$$\left(\begin{array}{ccccc|c}1&1&2&0&1&0\\-2&-1&1&-3&0&0\\3&2&1&3&1&0\\4&2&-2&6&0&0\end{array}\right) \xrightarrow[\substack{R_3-3R_1\\R_4-4R_1}]{R_2+2R_1} \left(\begin{array}{ccccc|c}1&1&2&0&1&0\\0&1&5&-3&2&0\\0&-1&-5&3&-2&0\\0&-2&-10&6&-4&0\end{array}\right)$$

$$\xrightarrow[R_4+2R_2]{\substack{R_1-R_2\\R_3+R_2}} \left(\begin{array}{ccccc|c}1&0&-3&3&-1&0\\0&1&5&-3&2&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)$$

$$\begin{array}{ccccc}&&s&t&u\end{array}$$

$\Rightarrow$ If $\underline{x} = (x_1, x_2, x_3, x_4, x_5)$ and $A\underline{x} = \underline{0}$

then
$x_1 = 3s - 3t + u$
$x_2 = -5s + 3t - 2u$
$x_3 = s$
$x_4 = t$
$x_5 = u$

and a basis is $\{(3,-5,1,0,0), (-3,3,0,1,0), (1,-2,0,0,1)\}$.

and the dimension is 3.

20. For each, determine whether the given set W is a subspace of $\mathbb{R}^3$. Justify your answer.

(a) $W = \{(x, y, z) \in \mathbb{R}^3 \mid xz = yz\}$ $\boxed{4}$

Not a subspace: Let $\underline{u} = (1,1,1)$ and $\underline{v} = (2,1,0)$

both $\underline{u}$ and $\underline{v}$ are in W

but $\underline{u}+\underline{v} = (3,2,1) \notin W$.

$\Rightarrow$ not closed under addition.

(b) $W = \{\mathbf{u} \in \mathbb{R}^3 \mid \mathbf{w} \cdot \mathbf{u} = 0\}$ for some fixed $\mathbf{w} \in \mathbb{R}^3$. $\boxed{4}$

Let $\underline{u}$ and $\underline{v} \in W$.

then $\underline{w} \cdot (\underline{u}+\underline{v}) = \underline{w} \cdot \underline{u} + \underline{w} \cdot \underline{v} = 0 + 0 = 0$

$\Rightarrow \underline{u}+\underline{v} \in W$.

Let $\underline{u} \in W$ and $k \in \mathbb{R}$.

then $\underline{w} \cdot (k\underline{u}) = k(\underline{w} \cdot \underline{u}) = k(0) = 0$

$\Rightarrow k\underline{u} \in W$.

$\Rightarrow W$ is a subspace.