STUDENT'S NAME: ______________________________

STUDENT'S NUMBER: ______________________________

DAWSON COLLEGE – DEPARTMENT OF MATHEMATICS
FINAL EXAM, WINTER 2014 **CALCULUS I (201-103-DW sections 01, 02, 03, 04)**
May 23, 2014 (2:00pm-5:00pm)
INSTRUCTOR(S): A. Douba, G. Honnouvo, A. Jimenez, N. Rossokhata

<u>INSTRUCTIONS</u>:

This exam has 14 pages.

No information sheet is provided

Make sure "NOW" that all the pages are included with your exam. If that is not the case, notify your teacher immediately

Do not detach any pages from this document. It should be returned intact.

Use "ONLY" the space provided for each one of the detailed answers. If additional space is required for an answer use the back of the sheet and clearly indicate its corresponding question

NO MARKS are given for missing or improperly labeled answers

Clearly write your name and student id in the space provided at the top of this page.

This exam has 12 questions for a total of 100 marks.

This exam is worth 40% of your final grade, that is, 40 marks

GOOD LUCK!

1.)

(12 Marks)

Find the value of the following limits. Write 'dne' when the limit does not exist.
Show all the work

a) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 5x + 6}$ $\frac{0}{0}$ - case => Factor (4 Marks)

$= \lim_{x \to 2} \frac{(x-1)\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} = \lim_{x \to 2} \frac{x-1}{x-3} = \frac{2-1}{2-3} = \frac{1}{-1} = -1$

b) $\lim_{x \to \infty} \frac{3x^5 + 2}{4x^5 - 5x^3 - 8}$ $= \frac{3}{4}$ (4 Marks)

c) $\lim_{x \to 3} \frac{x - 3}{\sqrt{x-2} - 1}$ $\frac{0}{0}$ - case => Rationalize/Factor (4 Marks)

$\lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)} = \lim_{x \to 3} \frac{\cancel{(x-3)}(\sqrt{x-2}+1)}{\cancel{x-2-1}}$

$= \sqrt{3-2} + 1 = 2$

2.) (6 Marks)
Given the following function

$$f(x)=\begin{cases} x^2-12 & if\ x<4 \\ 4 & if\ x=4 \\ \sqrt{1+2x} & if\ x>4 \end{cases}$$

a) Find $\lim_{x \to 4-} f(x) = \lim_{x \to 4} (x^2-12)$ (2 marks)

$= 4^2-12 = 4$

b) Find $\lim_{x \to 4+} f(x) = \lim_{x \to 4} \sqrt{1+2x}$ (2 marks)

$= \sqrt{1+2 \cdot 4} = 3$

c) Is the function continuous at x = 4?
Justify your claim and show all the work (2 marks)

No. $\boxed{4 = \lim_{x \to 4-} f(x) \neq \lim_{x \to 4+} f(x) = 3}$

3.) (6 Marks)
Find the derivative of f(x) using **only** the limit definition
(No marks are given for using any of the differentiation rules)

$$f(x)=2x^2+5x$$

$$f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} =$$

$$= \lim_{t \to x} \frac{2(t^2-x^2)+5(t-x)}{t-x}$$

$$= \lim_{t \to x} \frac{\cancel{(t-x)}(2(t+x)+5)}{\cancel{t-x}}$$

$$= 2(x+x)+5 = 4x+5$$

4) (4 marks)
Find the **second** derivative f''(x) of the function f(x) given by

$$f(x)=e^{2x}-\ln(x^2)+\sin(x)$$

$$f'(x) = 2e^{2x} - \frac{2}{x} + \cos(x)$$

$$f''(x) = 4e^{2x} + \frac{2}{x^2} - \sin(x)$$

5.)
(7 Marks)
Let y be the function given implicitly by $x^3 - y^3 = 6xy$

a) Verify that the point (-3, 3) is on the funtion (1 mark)

$(-3)^3 - (3)^3 = 6(-3)(3)$

$-2 \cdot 3^3 = -2 \cdot 3^3$

b) Calculate the first derivative of y (that is, y') implicitly (4 marks)

$3x^2 - 3y^2 y' = 6y + 6xy'$

$$\boxed{y' = \frac{3x^2 - 6y}{6x + 3y^2}}$$

c) Find the equation of the tangent line to y at (-3, 3) (2 marks)

$$m = y'(-3) = \frac{3(-3)^2 - 6 \cdot 3}{6(-3) + 3 \cdot 3^2} = \frac{9}{9} = 1$$

$3 = -3 + b \Rightarrow b = 6$

$$\boxed{y = x + 6}$$

6.)

(6 Marks)
Find the absolute maximum and absolute minimum of the function

$$f(x)=2x^3-3x^2-12x$$

on the closed interval [-2, 4]

$$0=f'(x)=6x^2-6x-12=6(x-2)(x+1)$$

$$x_1=-1 \qquad x_2=2$$

$$f(-2)=2(-2)^3-3(-2)^2-12(-2)=-4$$

$$f(-1)=2(-1)^3-3(-1)^2-12(-1)=7$$

$$f(2)=2\cdot 2^3-3\cdot 2^2-12\cdot 2=-20 \quad \underline{min}$$

$$f(4)=2\cdot 4^3-3\cdot 4^2-12\cdot 4=5\cdot 4^2-12\cdot 4=32$$

$\underline{Max}$

7.)
(4 marks)

Find the point(s) on the graph of the function $f(x)=\frac{x^2}{x-1}$ where the tangent line is horizontal.

$$f'(x)=\frac{2x(x-1)-x^2}{(x-1)^2}=0$$

$$x^2-2x=0 \quad \Rightarrow \quad \boxed{\begin{array}{l} x_1=0 \\ x_2=2 \end{array}}$$

8.)
(17 marks)

Let $f(x) = x^3 + 3x^2 - 4$

a) Show that -2 and 1 are the x-intercepts of this function (1 mark)

$f(-2) = (-2)^3 + 3(-2)^2 - 4 = 0$

$f(1) = 1^3 + 3 \cdot 1^2 - 4 = 0$

b) Find the interval(s) on which f is increasing and the interval(s) on which it is increasing (4 marks)

$f'(x) = 3x^2 + 6x = 3x(x+2) = 0$

$f'(-3) > 0$ ↗ inc

$\boxed{(-\infty,-2) \cup (0,\infty) \text{ inc}}$

Max $x_1 = -2$

$f'(-1) < 0$ ↘ dec

$\boxed{(-2,0) \text{ dec}}$

min $x_2 = 0$

$f'(1) > 0$ ↗ inc

c) Find the x and y coordinates of the relative extrema of f (if any) (2 marks)

$f(-2) = 0$ Max

$f(0) = -4$ min

d) Find the interval(s) on which the function is concave upward and the interval(s) where it is concave downward (3 marks)

$f''(x) = 6x + 6 = 6(x+1) = 0$

$f''(-2) < 0$ ↓ down

$x = -1$

$(-\infty, -1)$ down

$f''(0) > 0$ ↑ up

$(-1, \infty)$ up

e) Find the x and y coordinates of the inflection point(s) (if any) (1 mark)

$f(-1) = 2(-1)^3 + 3(-1)^2 - 4 = -3$

inflection point

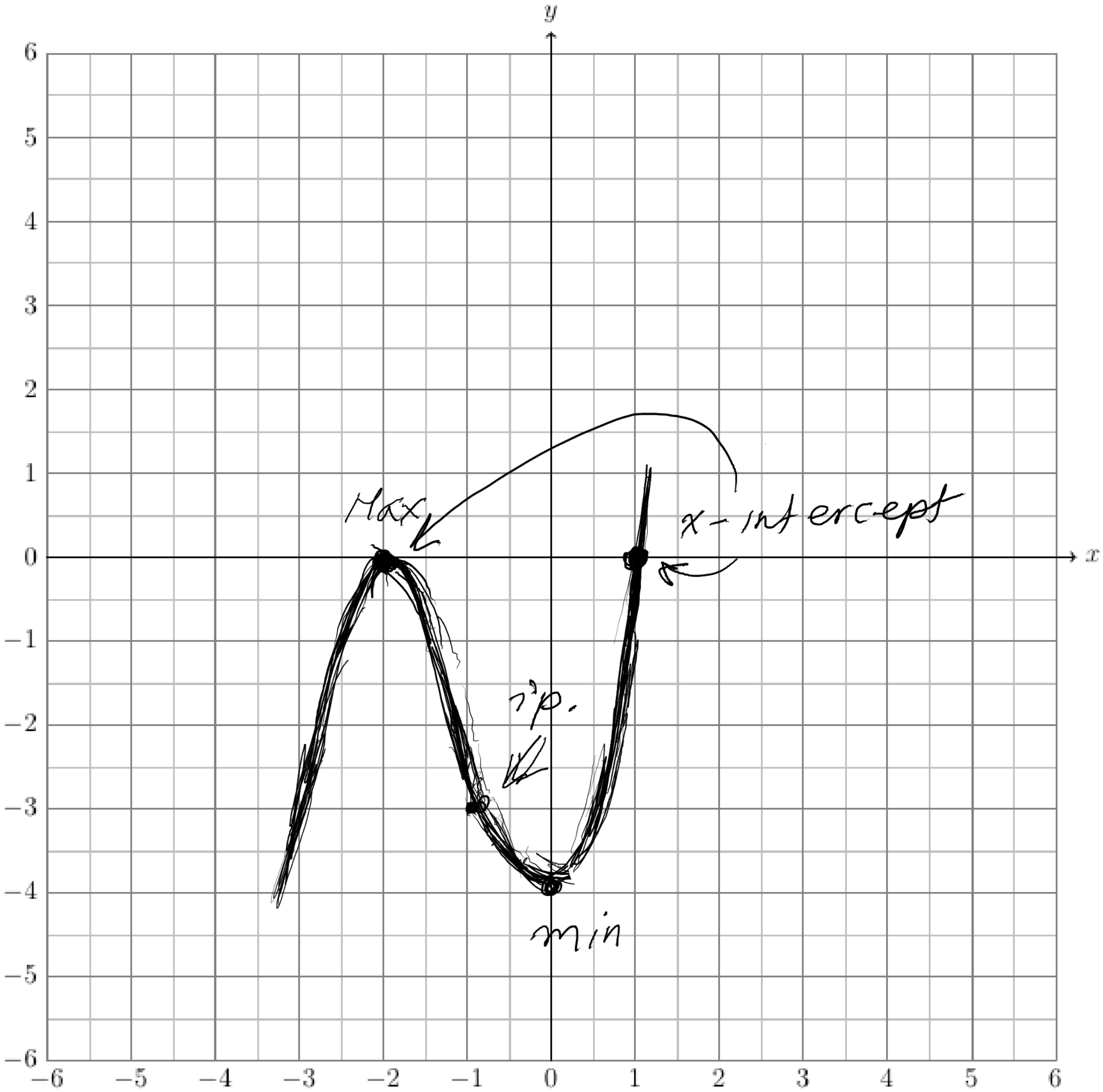
f) Find the $\lim_{x\to-\infty} f(x)$ and the $\lim_{x\to\infty} f(x)$ (2 marks)

$\lim_{x\to-\infty} f(x) = -\infty$

$\lim_{x\to\infty} f(x) = \infty$

g) On the grid below sketch the graph of f(x) indicating the intercepts and any relative maxima, relative minima and inflection point(s) you have found

(4 marks)



9.)
(12 Marks)
The total annual cost (in dollars) incurred by a company for manucfacturing the first x units of a product is given by $C(x)=0.001x^2+300x+350000$

a) Find the actual cost incurred in manucfacturing the 101-st unit (3 marks)

$C(101)-C(100)=$

$0.001(101^2-100^2)+300(101-100)=$

$0.001\times 201+300=300.20$ (dollars)

b) Use only the marginal cost to estimate the cost incurred in manucfacturing the 250-th unit (3 Marks)

$C'(x)=0.002x+300$

$C'(249)=0.002(249)+300\approx 300.50$

c) If the demand function (unit price) is given by $p(x)=720-0.004x$ find the revenue function and (2 Marks)

$R(x)=-0.004x^2+720x$

d) Find the level of production (x) that maximizes the profit (4 marks)

$P(x)=R(x)-C(x)=-0.005x^2+420x-350000$

$P'(x)=-0.01x+420=0$

$x=\dfrac{420}{0.01}=42000$

10.)
(12 marks)
Calculate the derivative of each one of the following functions using any one of the applicable differentiation rules. **Do not simplify**

a) $y(x)=(x^2+5)^{\sin(x)}$ (Use logarithmic differentiation) (3 Marks)

$$y'(x) = (x^2+5)^{\sin(x)}\left([\sin(x)]'\ln(x^2+5) + \sin(x)[\ln(x^2+5)]'\right)$$

$$= (x^2+5)^{\sin(y)}\left(\cos(x)\ln(x^2+5) + \sin(x)\frac{2x}{x^2+5}\right)$$

b) $f(x)=\ln\frac{(x^3+1)^4(x^4+1)^3}{\sqrt{x-6}}$ (3 Marks)

$$f'(x) = \frac{4\cdot 3x^2}{x^3+1} + \frac{3\cdot 4x^3}{x^4+1} - \frac{1}{2}\frac{1}{x-6}$$

c) $g(x) = x^2 \ln(\tan(x))$ (3 Marks)

$$g'(x) = 2x \ln(\tan(x) + x^2 \frac{\sec^2(x)}{\tan(x)}$$

d) $h(x) = \frac{\cos(x^2)}{x^3 - 1}$ (3 marks)

$$h'(x) = \frac{-\sin(x^2)2x(x^3-1) - \cos(x^2)3x^2}{(x^3-1)^2}$$

11.)
(8 marks)

Each edge of a square piece of cardboard is 6 inches long. By cutting away identical squares from each corner (as shown in the diagram) and folding up the resulting flaps, an open box is made. Find the dimensions of the box yielding the maximum volume

$V(x) = x(6-2x)^2$

$V'(x) = [36x - 24x^2 + 4x^3]'$

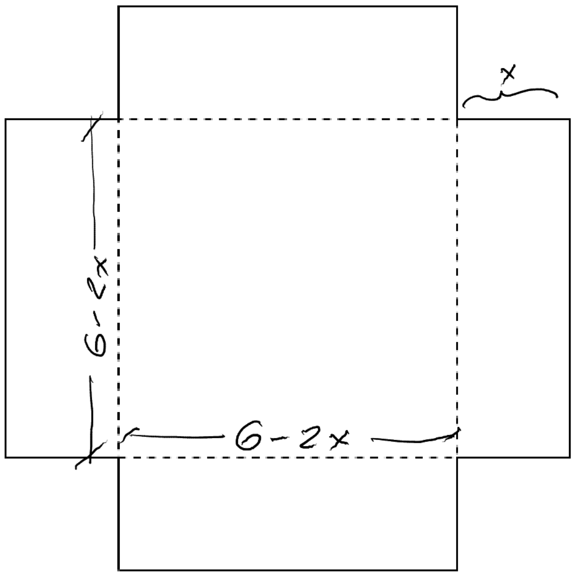
$= 12x^2 - 48x + 36 =$

$12(x^2 - 4x + 3) = 12(x-3)(x-1) = 0$

$\Rightarrow \boxed{x = 1}$ $\boxed{x = 3}$ discarded

$\checkmark$

$\boxed{\text{Dimensions } 1 \times 4 \times 4}$



12.)
(6 Marks)
Find the following antiderivatives. **Do not simplify**

a) $\int (\sqrt[3]{x^2} - \frac{1}{x^2} + 8)\,dx$ (3 Marks)

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} - \frac{x^{-2+1}}{-2+1} + 8x + C$$

$$= \frac{3}{5}\sqrt[3]{x^5} + \frac{1}{x} + 8x + C$$

b) $\int \frac{5x-3}{\sqrt{5x^2-6x+2}}\,dx = \int \frac{\cancel{5x-3}}{\sqrt{u}} \frac{1}{2(\cancel{5x-3})} du =$ (3 Marks)

$$u = 5x^2 - 6x + 2$$
$$u' = 10x - 6 = 2(5x-3)$$
$$dx = \frac{1}{2(5x-3)} du$$

$$\frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \sqrt{5x^2 - 6x + 2} + C$$