DAWSON COLLEGE
MATHEMATICS DEPARTMENT
Linear Algebra (SCIENCE)

201-NYC-05 S01-S07, S11
Winter 2017
Final Examination
May 25th, 2017
Time Limit: 3 hours

Name: ______________________

ID#: ______________________

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- This exam contains 8 pages (including this cover page) and 18 problems. Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your solutions in a legible and orderly manner.
- You are only permitted to use the **Sharp EL-531XG** or **Sharp EL-531X** calculator.
- This examination booklet must be returned intact.
- Good luck!

Question	Points	Score
1	9	
2	5	
3	3	
4	5	
5	5	
6	5	
7	4	
8	4	
9	4	
10	3	
11	5	
12	5	
13	5	
14	5	
15	4	
16	6	
17	8	
18	15	
Total:	100	

1. Given

$$M = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where M is in row echelon form.

(a) (3 marks) Find the reduced row echelon form of M.

Answer:
$$\begin{bmatrix} 1 & 2 & 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (3 marks) Find the solution set of the system of linear equations whose augmented matrix is M.

Answer:
$(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (-2-2s+3t,\ s,\ -t,\ t,\ 1)$ where $s,t \in \mathbb{R}$

(c) (3 marks) Find a basis for the solution space of the homogeneous system of linear equations whose coefficient matrix is M.

Answer:
$B = \{(2,\ 0,\ 0,\ 0,\ -1,\ 1),\ (3,\ 0,\ -1,\ 1,\ 0,\ 0),\ (-2,\ 1,\ 0,\ 0,\ 0,\ 0)\}$

2. (5 marks) Consider the system

$$\begin{array}{rcrcrcl} 2kx & + & (k+1)y & & & = & 2 \\ x & + & y & + & z & = & 0 \\ kx & + & (2k-1)y & & & = & 1 \end{array}$$

Find the value(s) of k, if any, such that the system has: a) no solutions, b) a unique solution, c) infinitely many solutions.

Answer:

a) $k = 0$
b) $k \neq 0$ or $k \neq 1$
c) $k = 1$

3. (3 marks) Evaluate:

$$\begin{bmatrix} 0 & 0 & \sqrt[4]{2} \\ 0 & 1 & 0 \\ \sqrt[4]{2} & 0 & 0 \end{bmatrix}^{24}$$

Answer:
$$\begin{bmatrix} 64 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{bmatrix}$$

4. (5 marks) Solve for A, if possible:

$$A^{-1}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} = \left(\begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix} - 2A\right)^{-1}$$

Answer:
$$A = \begin{bmatrix} 1 & 0 \\ 7 & 3 \end{bmatrix}$$

5. (5 marks) Express

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 3 & 3 \end{bmatrix}$$

as a product of elementary matrices, E_i, and a reduced row echelon matrix, R. That is, express A as $E_k \cdots E_2 E_1 R$.

Answer:
$$A = E_3E_2E_1R = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

6. (5 marks) Given the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -2 & 6 \\ 0 & 1 & -3 \end{bmatrix}$$

Find $\det(X)$ given that X satisfies the equation $(X+BC)^{-1} = A$.

Answer:
$\det(X) = 8$

7. (4 marks) Given $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -2$, find $\begin{vmatrix} 3c-d & 6a-2b \\ 2d & 4b \end{vmatrix}$.

Answer:
24

8. (4 marks) Given $A = \begin{bmatrix} x^2 & 1 & x \\ 3 & 1 & 2 \\ x & -1 & -1 \end{bmatrix}$ find the value(s) of x, if any, such that $\det(A) = 0$.

Answer:
$x = 3$

9. (4 marks) Given A, an $n \times n$ matrix such that $\det(A) = 9$ and

$$A^3 A^T = 3A^{-1}\text{adj}(A)$$

find n.

Answer:
$n = 4$

10. (3 marks) Find all vectors of length 2 that are orthogonal to both $\vec{u} = (-2,3,1)$ and $\vec{v} = (1,2,-3)$.

Answer:
$\frac{\pm 2}{\sqrt{195}}(-11,\ -5,\ -7)$

11. (5 marks) Given the lines:

$$\begin{aligned} \mathscr{L}_1 &: (x,y,z) = (1,2,-2) + t_1(1,2,1) \\ \mathscr{L}_2 &: (x,y,z) = (2,1,3) + t_2(1,2,3) \\ \mathscr{L}_3 &: (x,y,z) = (1,1,1) + t_3(2,7,3) \text{ where } t_1,\ t_2,\ t_3 \in \mathbb{R}. \end{aligned}$$

Find the equation of the line which is parallel to $\mathscr{L}_3$ and which intersects both $\mathscr{L}_1$ and $\mathscr{L}_2$.

Answer:
$(x,\ y,\ z) = (\frac{3}{2},\ 3,\ -\frac{3}{2}) + t(2,\ 7,\ 3)$ where $t \in \mathbb{R}$

12. (5 marks) Given the non-intersecting lines:

$$\begin{aligned} \mathscr{L}_1 &: (x,y,z) = (1,2,-2) + t_1(1,2,1) \\ \mathscr{L}_2 &: (x,y,z) = (2,1,3) + t_2(1,2,3) \text{ where } t_1,\ t_2 \in \mathbb{R}. \end{aligned}$$

Find the shortest distance between $\mathscr{L}_1$ and $\mathscr{L}_2$.

Answer:
$\frac{3}{5}\sqrt{5}$

13. (5 marks) Find the closest point on the plane $2x+y-3z=1$ to the point $P(3,26,1)$.

Answer:
$(-1,\ 24,\ 7)$

14. (5 marks) Given the points $A(1,2,-1)$ and $B(1,1,2)$. Find the point C on the y-axis such that the area of the triangle ABC is $\frac{\sqrt{10}}{2}$.

Answer:
$C = \left(0,\ \frac{5}{3},\ 0\right)$

15. (4 marks) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be non-zero vectors in $\mathbb{R}^3$. Show that if $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} = 0$ then $\{\vec{u},\vec{v},\vec{w}\}$ is a basis for $\mathbb{R}^3$.

Answer:
Since $\dim(\mathbb{R}^3) = 3$ then it is sufficient to show that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly indpendent to conclude that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ spans $\mathbb{R}^3$, hence is a basis of $\mathbb{R}^3$.

Given $c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=\vec{0}$, and taking the dot product of both sides of the equality with $\vec{u}$.

$$\begin{aligned}
\vec{u}\cdot(c_1\vec{u}+c_2\vec{v}+c_3\vec{w}) &= \vec{u}\cdot\vec{0}\\
\vec{u}\cdot(c_1\vec{u})+\vec{u}\cdot(c_2\vec{v})+\vec{u}\cdot(c_3\vec{w}) &= 0\\
c_1\vec{u}\cdot\vec{u}+c_2\vec{u}\cdot\vec{v}+c_3\vec{u}\cdot\vec{w} &= 0 \text{ since } \vec{u}\cdot\vec{v}=\vec{u}\cdot\vec{w}=0\\
c_1||\vec{u}||^2 &= 0 \text{ since } \vec{u}\neq\vec{0}\\
c_1 &= 0
\end{aligned}$$

is obtained. Similarly if applied using $\vec{v}$ and $\vec{w}$ then $c_2=0$ and $c_3=0$ is obtained. Since only the trivial solution satisfies the linear combination that gives the zero vector, $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent and is a basis of $\mathbb{R}^3$.

16. Given $V=\{a_0+a_1x+a_2x^2 \mid a_0+a_1=\pi \text{ and } a_i\in\mathbb{R}\}$ and the following vector addition and scalar multiplication:

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1-\pi)x+a_2b_2x^2 \text{ and } r\cdot p(x)=ra_0+(ra_1-\pi)x+(a_2)^rx^2$$

where $p(x)=a_0+a_1x+a_2x^2$ and $q(x)=b_0+b_1x+b_2x^2$.

(a) (2 marks) Determine whether the above set is closed under vector addition.

Answer:
Closed under vector addition.

(b) (2 marks) Determine whether the above set is closed under scalar multiplication.

Answer:
Not closed under scalar multiplication.

(c) (2 marks) Is V a vector space under the given operations? Justify

Answer:
Not a vector space since at least one of the ten axioms fail. Namely, closure under scalar multiplication.

17. Given $B=\{M_1,\ M_2,\ M_3\}$ a basis of $W=\text{span}(B)$ where $M_1=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $M_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $M_3=\begin{bmatrix}1&2\\0&-1\end{bmatrix}$.

(a) (1 mark) State the $\dim(W)$.

Answer:
$\dim(W)=3$

(b) (2 marks) Is $\begin{bmatrix}0&0\\0&0\end{bmatrix}\in W$? Justify.

Answer:
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$, since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1M_1 + c_2M_2 + c_3M_3$ is satisfied when $c_1 = c_2 = c_3 = 0$.

(c) (2 marks) Is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in W$? Justify.

Answer:
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin W$ since $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = c_1M_1 + c_2M_2 + c_3M_3$ gives rise to the inconsistent equation $c_1 0 + c_2 0 + c_3 0 = 1$.

(d) (3 marks) Show $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \in W$ and find the coordinates of M relative to the basis B. That is, find $(M)_B$.

Answer:
$M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = c_1M_1 + c_2M_2 + c_3M_3$ is satisfied when $c_1 = 2$, $c_2 = 0$ and $c_3 = -1$, hence $(M)_B = (2, 0, -1)$.

18. Determine whether the following statements are true or false for any $n \times n$ matrices A and B. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

(a) (3 marks) If $A^2 = 0$ then $A = 0$.

Answer:
False, if $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $A^2 = 0$ but $A \neq 0$.

(b) (3 marks) If A is symmetric and skew-symmetric[1] then $A = 0$.

Answer:
True, by the premises $A^T = A$ and $A^T = -A$, it follows that

$$\begin{aligned} A^T &= A^T \\ A &= -A \\ 2A &= 0 \\ A &= 0 \end{aligned}$$

(c) (3 marks) If $AA^T = A$ then A is symmetric.

Answer:
True, since $A^T = (AA^T)^T = (A^T)^T A^T = AA^T = A$.

[1] A is said to be *skew-symmetric* if $A^T = -A$.

(d) (3 marks) If A^2 is an elementary matrix then A is an elementary matrix.

Answer:
False, if $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ then $A^2 = I$ which is an elementary matrix but A is not an elementary matrix.

(e) (3 marks) If A and B are invertible then A and B are row equivalent.

Answer:
True, the reduced row echelon form of A and B is I. Hence there exists a finite sequence of elementary row operations that when applied on A result in I. And a second finite sequence of elementary row operations that when applied on B result in I. If the first sequence of elementary row operations is applied on A followed by the inverse elementary row operations applied in reverse order of the second sequence of elemtary row operations then B is obtained. Hence A and B are row equivalent.