Dawson College

Mathematics Department

FINAL EXAMINATION FOR WINTER 2019

CALCULUS I (Electronics Engineering Technology)

201-NYA-05 section 7

May 27, 2019 9:30am - 12:30pm

Student Name: Solutions

Student I.D. #: ______________________

Instructor: Alexander Hariton

Time: 3 hours

Instructions:

- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.
- A Sharp EL-531XG, Sharp EL-531XT or Sharp EL-531X calculator is permitted.

This examination consists of 12 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. **(16 marks)**

Evaluate the following limits. Show your work.

(a) $\displaystyle\lim_{x\to -5} \frac{\sqrt{1-3x}-4}{x+5}$

$$= \lim_{x\to -5} \frac{\sqrt{1-3x}-4}{x+5} \cdot \frac{\sqrt{1-3x}+4}{\sqrt{1-3x}+4}$$

$$= \lim_{x\to -5} \frac{(1-3x)-16}{(x+5)(\sqrt{1-3x}+4)} = \lim_{x\to -5} \frac{-3x-15}{(x+5)(\sqrt{1-3x}+4)}$$

$$= \lim_{x\to -5} \frac{-3(x+5)}{(x+5)(\sqrt{1-3x}+4)} = \lim_{x\to -5} \frac{-3}{\sqrt{1-3x}+4}$$

$$= \frac{-3}{\sqrt{1-3(-5)}+4} = \frac{-3}{\sqrt{16}+4} = \frac{-3}{4+4} = \boxed{-\frac{3}{8}}$$

(b) $\displaystyle\lim_{x\to 3} \frac{2x^2-5x-3}{x^2-9}$

$$2x^2-5x-3 = 2x^2-6x+x-3 = 2x(x-3)+(x-3) = (2x+1)(x-3)$$

so $$\lim_{x\to 3} \frac{2x^2-5x-3}{x^2-9} = \lim_{x\to 3} \frac{(2x+1)(x-3)}{(x+3)(x-3)}$$

$$= \lim_{x\to 3} \frac{2x+1}{x+3} = \frac{2(3)+1}{3+3} = \boxed{\frac{7}{6}}$$

1. (continued)

(c) $\displaystyle\lim_{x\to\infty} \frac{x^4 - 2x^3 + 8}{x^2(5 - x^2)}$

$$= \lim_{x\to\infty} \frac{x^4 - 2x^3 + 8}{5x^2 - x^4} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)}$$

$$= \lim_{x\to\infty} \frac{1 - \frac{2}{x} + \frac{8}{x^4}}{\frac{5}{x^2} - 1} = \frac{1 - 0 + 0}{0 - 1} = \boxed{-1}$$

(d) $\displaystyle\lim_{x\to 2^-} \frac{3x^2}{x^2 - x - 2}$

$$\frac{3x^2}{x^2 - x - 2} = \frac{3x^2}{(x-2)(x+1)}$$

As $x \to 2^-$, $x < 2$, so $x - 2 < 0$. so $x - 2 \to 0^-$

Also, $3x^2 \to 3(2)^2 = 12$, and $x + 1 \to 2 + 1 = 3$

so $$\frac{3x^2}{(x-2)(x+1)} \to \frac{12}{(0^-)(3)} \to -\infty$$

so $$\lim_{x\to 2^-} \frac{3x^2}{x^2 - x - 2} = \boxed{-\infty}$$

2. (4 marks)

Let

$$f(x) = \begin{cases} \frac{2}{x+1} & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

For what values of x is $f(x)$ not continuous? Justify your answer by referring to the definition of continuity.

on interval $(-\infty, 1)$, $f(x)$ is rational, so it is continuous everywhere except where $x+1=0$, that is, $x=-1$

At $x=-1$, f is not defined, so it is not continuous.

On interval $(1, \infty)$, $f(x)$ is polynomial, so it is continuous everywhere.

At $x = 1$, $f(1) = 3$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\frac{2}{x+1}\right) = \frac{2}{1+1} = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 2(1)-1 = 1$$

so $\lim_{x \to 1} f(x) = 1$, but $f(1) = 3$

so $\lim_{x \to 1} f(x) \neq f(1)$, thus f is not continuous at $x=1$.

So f is not continuous at $\boxed{x=-1}$ and at $\boxed{x=1}$

3. **(16 marks)**

Find the derivatives of the following functions. Answers do not need to be simplified, but should be expressed only in terms of x (not in terms of other derivatives).

(a) $f(x) = e^{(x^2+1)} \tan(3x-4)$

$$f'(x) = \frac{d}{dx}\left[e^{(x^2+1)}\right] \cdot \tan(3x-4) + e^{(x^2+1)} \cdot \frac{d}{dx}\left[\tan(3x-4)\right]$$

$$= e^{(x^2+1)} \cdot \frac{d}{dx}(x^2+1) \cdot \tan(3x-4)$$

$$+ e^{(x^2+1)} \cdot \sec^2(3x-4) \cdot \frac{d}{dx}(3x-4)$$

$$= \boxed{e^{(x^2+1)}(2x)\tan(3x-4) + e^{(x^2+1)}\sec^2(3x-4)(3)}$$

(b) $f(x) = \sqrt[3]{x} + x^2 \sin^{-1}(2x) - 5^x + 7$

$$f'(x) = \frac{d}{dx}\left(x^{1/3}\right) + \frac{d}{dx}\left(x^2 \sin^{-1}(2x)\right) - \frac{d}{dx}(5^x) + 0$$

$$= \frac{1}{3}x^{-2/3} + \frac{d}{dx}(x^2) \cdot \sin^{-1}(2x) + x^2 \cdot \frac{d}{dx}\left(\sin^{-1}(2x)\right)$$

$$- 5^x \ln 5$$

$$= \frac{1}{3}x^{-2/3} + 2x\sin^{-1}(2x) + x^2 \cdot \left(\frac{1}{\sqrt{1-(2x)^2}}\right) \cdot (2) - 5^x \ln 5$$

$$= \boxed{\frac{1}{3x^{2/3}} + 2x\sin^{-1}(2x) + \frac{2x^2}{\sqrt{1-4x^2}} - 5^x \ln 5}$$

3. (continued)

(c) $f(x) = \left(\dfrac{2x-1}{2x+1}\right)^6$

$$f'(x) = 6\left(\frac{2x-1}{2x+1}\right)^5 \cdot \frac{d}{dx}\left(\frac{2x-1}{2x+1}\right)$$

$$= 6\left(\frac{2x-1}{2x+1}\right)^5 \cdot \left[\frac{\frac{d}{dx}(2x-1)\cdot(2x+1) - (2x-1)\cdot\frac{d}{dx}(2x+1)}{(2x+1)^2}\right]$$

$$= \boxed{6\left(\frac{2x-1}{2x+1}\right)^5 \left[\frac{2(2x+1) - 2(2x-1)}{(2x+1)^2}\right]}$$

(d) $f(x) = \cos\left(e^{\sqrt{x^3+1}}\right)$

$$f'(x) = -\sin\left(e^{\sqrt{x^3+1}}\right) \cdot \frac{d}{dx}\left(e^{\sqrt{x^3+1}}\right)$$

$$= -\sin\left(e^{\sqrt{x^3+1}}\right) \cdot e^{\sqrt{x^3+1}} \cdot \frac{d}{dx}\left(\sqrt{x^3+1}\right)$$

$$= -\sin\left(e^{\sqrt{x^3+1}}\right) \cdot e^{\sqrt{x^3+1}} \cdot \frac{1}{2\sqrt{x^3+1}} \cdot \frac{d}{dx}(x^3+1)$$

$$= \boxed{-\sin\left(e^{\sqrt{x^3+1}}\right) \cdot e^{\sqrt{x^3+1}} \cdot \frac{1}{2\sqrt{x^3+1}} \cdot (3x^2)}$$

4. (6 marks)

Find the equations of the tangent and normal lines to the curve $f(x) = \dfrac{2x^2}{2 - \ln x}$ at the point $(1, 1)$

$$f'(x) = \frac{\frac{d}{dx}(2x^2)\cdot(2-\ln x) - 2x^2\cdot\frac{d}{dx}(2-\ln x)}{(2-\ln x)^2}$$

$$= \frac{4x(2-\ln x) - 2x^2\left(-\frac{1}{x}\right)}{(2-\ln x)^2} = \frac{8x - 4x\ln x + 2x}{(2-\ln x)^2}$$

$$= \frac{10x - 4x\ln x}{(2-\ln x)^2} = \frac{2x(5-2\ln x)}{(2-\ln x)^2}$$

At point $(1,1)$, $x=1$, so the slope of the tangent line is:

$$m_t = f'(1) = \frac{2(1)(5-2\ln(1))}{(2-\ln(1))^2} = \frac{2(5-0)}{(2-0)^2} = \frac{10}{4} = \frac{5}{2}$$

The equation of the tangent is: $y-1 = \frac{5}{2}(x-1)$, so

$y-1 = \frac{5}{2}x - \frac{5}{2}$, so $\boxed{y = \frac{5}{2}x - \frac{3}{2}}$

The slope of the normal is: $m_n = -\dfrac{1}{m_t} = -\dfrac{1}{\left(\frac{5}{2}\right)} = -\dfrac{2}{5}$

The equation of the normal is: $y-1 = -\frac{2}{5}(x-1)$, so

$y-1 = -\frac{2}{5}x + \frac{2}{5}$, so $\boxed{y = -\frac{2}{5}x + \frac{7}{5}}$

5. (8 marks)

(a) For the curve defined by the equation $x^3 - x \sin y + 2y^3 = 5$, find $\dfrac{dy}{dx}$

$$\frac{d}{dx}\left(x^3 - x\sin y + 2y^3\right) = \frac{d}{dx}(5)$$

$$3x^2 - \frac{d}{dx}(x \sin y) + 2(3y^2)\frac{dy}{dx} = 0$$

$$3x^2 - \left[(1)\sin y + x\cos y \frac{dy}{dx}\right] + 6y^2\frac{dy}{dx} = 0$$

$$3x^2 - \sin y - x\cos y\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$$

$$3x^2 - \sin y = x\cos y\frac{dy}{dx} - 6y^2\frac{dy}{dx}$$

$$3x^2 - \sin y = (x\cos y - 6y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \boxed{\frac{3x^2 - \sin y}{x\cos y - 6y^2}}$$

(b) Use logarithmic differentiation to find the derivative of the function

$f(x) = (x+2)^{\cos x}$

Let $y = (x+2)^{\cos x}$, then $\ln y = \cos x \ln(x+2)$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\cos x \ln(x+2)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\cos x)\cdot\ln(x+2) + \cos x \cdot \frac{d}{dx}\left[\ln(x+2)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(x+2) + \cos x\left(\frac{1}{x+2}\right)(1)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x+2} - \sin x \ln(x+2)\right]$$

so $$f'(x) = \boxed{(x+2)^{\cos x}\left[\frac{\cos x}{x+2} - \sin x\ln(x+2)\right]}$$

6. (5 marks)

If the function $f(x) = 2x^3 - 6x^2 + 1$ is restricted to the closed interval $[-2, 1]$, find the absolute maximum and absolute minimum of $f(x)$ on $[-2, 1]$

$$f'(x) = 6x^2 - 12x = 6x(x-2)$$

so $f'(x) = 0$ if $\underline{x=0}$ or $\underline{x=2}$ and $f'(x)$ exists everywhere.

The critical numbers are 0 and 2, of which only 0 is in the interval $[-2, 1]$

$$f(0) = 2(0)^3 - 6(0)^2 + 1 = 1$$

At the endpoints:

$$f(-2) = 2(-2)^3 - 6(-2)^2 + 1 = -16 - 24 + 1 = -39$$

$$f(1) = 2(1)^3 - 6(1)^2 + 1 = 2 - 6 + 1 = -3$$

so the absolute maximum is $\boxed{1}$, occurring at $x=0$

and the absolute minimum is $\boxed{-39}$, occurring at $x=-2$.

7. (12 marks)

For the function $f(x) = \dfrac{2x}{(x+1)^2}$ find:

(a) the x and y intercepts (if any)

(b) the horizontal and vertical asymptotes (if any)

(c) the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing

(d) the relative maxima and minima (if any)

(e) the intervals where $f(x)$ is concave up and where $f(x)$ is concave down

(f) the points of inflection (if any)

Use the above information to sketch the graph of the function $f(x)$

Note: $f'(x) = \dfrac{2(1-x)}{(x+1)^3}$ $\quad f''(x) = \dfrac{4(x-2)}{(x+1)^4}$

(a) y-intercept : let $x=0$: $f(0) = \dfrac{2(0)}{(0+1)^2} = \dfrac{0}{1^2} = 0$

so $\boxed{(0,0)}$ is the y-intercept.

x-intercepts : let $f(x) = 0$: $\dfrac{2x}{(x+1)^2} = 0$, so $2x=0$, $x=0$

so $\boxed{(0,0)}$ is the only x-intercept.

(b) horizontal asymptotes: $f(x) = \dfrac{2x}{x^2+2x+1} = \dfrac{\frac{2}{x}}{1+\frac{2}{x}+\frac{1}{x^2}}$

so $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\frac{2}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = \dfrac{0}{1+0+0} = \dfrac{0}{1} = 0$

and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{\frac{2}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = \dfrac{0}{1+0+0} = \dfrac{0}{1} = 0$

so $\boxed{y=0}$ is a horizontal asymptote

vertical asymptotes: can occur only where $(x+1)^2=0$: $x+1=0$: $x=-1$

$\lim\limits_{x\to-1^-} f(x) = \lim\limits_{x\to-1^-} \dfrac{2x}{(x+1)^2} \to \dfrac{2(-1)}{0^+} \to -\infty$: $\boxed{\lim\limits_{x\to-1^-} f(x) = -\infty}$

$\lim\limits_{x\to-1^+} f(x) = \lim\limits_{x\to-1^+} \dfrac{2x}{(x+1)^2} \to \dfrac{2(-1)}{0^+} \to -\infty$: $\boxed{\lim\limits_{x\to-1^+} f(x) = -\infty}$

7. (continued)

so $\boxed{x=-1}$ is a vertical asymptote.

(c) $f'(x) = \dfrac{2(1-x)}{(x+1)^3}$, so $f'(x)=0$ if $\underline{x=1}$ and $f'(x)$ does not exist if $\underline{x=-1}$

Interval	2	$1-x$	$(x+1)^3$	$f'(x)$	f
$x<-1$	+	+	−	−	↘
$-1<x<1$	+	+	+	+	↗
$x>1$	+	−	+	−	↘

f is increasing on $(-1,1)$ and decreasing on $(-\infty,-1)$ and $(1,\infty)$.

(d) There is a relative maximum at $x=1$: $f(1) = \dfrac{2(1)}{(1+1)^2} = \dfrac{1}{2}$

So $\boxed{(1, \tfrac{1}{2})}$ is a relative maximum.

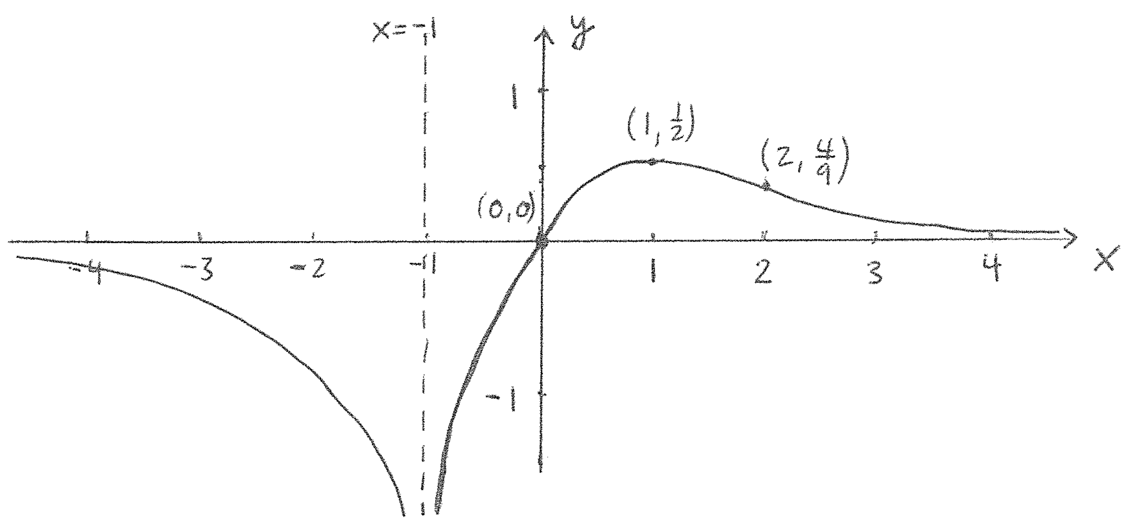
(e) $f''(x) = \dfrac{4(x-2)}{(x+1)^4}$, so $f''(x)=0$ if $\underline{x=2}$ and $f''(x)$ does not exist if $\underline{x=-1}$

Interval	4	$x-2$	$(x+1)^4$	$f''(x)$	f
$x<-1$	+	−	+	−	∩
$-1<x<2$	+	−	+	−	∩
$x>2$	+	+	+	+	∪

f is concave up on $(2,\infty)$ and concave down on $(-\infty,-1)$ and $(-1,2)$

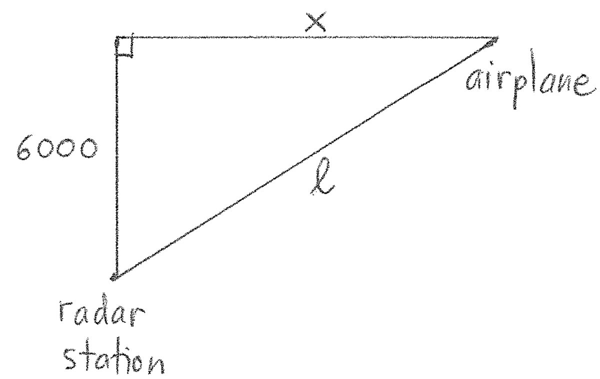
(f) There is a point of inflection at $x=2$: $f(2) = \dfrac{2(2)}{(2+1)^2} = \dfrac{4}{9}$

so $\boxed{(2, \tfrac{4}{9})}$ is a point of inflection



8. (6 marks)

An airplane is flying horizontally at an altitude of 6000 m, at a (constant) speed of 200 m/s. At a certain moment, the airplane flies directly over a radar station. At a time 40 seconds later, find the rate at which the distance between the station and the airplane is increasing.



Let ℓ be the distance (in m) between the airplane and the radar station.

Let x be the horizontal distance (in m) between the airplane and the point directly (6000 m) above the radar station.

$\frac{dx}{dt} = 200$ m/s , $\frac{d\ell}{dt}$ is to be determined.

Pythagorean Theorem: $\ell^2 = x^2 + (6000)^2$

$$2\ell \frac{d\ell}{dt} = 2x \frac{dx}{dt} + 0$$

$$\frac{d\ell}{dt} = \frac{2x \frac{dx}{dt}}{2\ell}, \quad \text{so} \quad \boxed{\frac{d\ell}{dt} = \frac{x}{\ell} \frac{dx}{dt}}$$

After 40 seconds, flying at 200 m/s, the airplane will travel:

$x = vt = (200 \text{ m/s})(40 \text{ s}) = \boxed{8000 \text{ m}}$

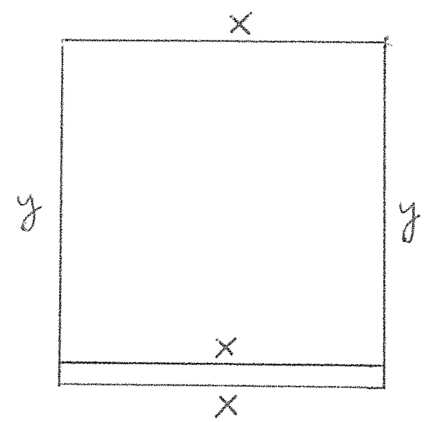
Also, $\ell = \sqrt{x^2 + (6000)^2} = \sqrt{(8000)^2 + (6000)^2} = \boxed{10\,000 \text{ m}}$

so $\frac{d\ell}{dt} = \frac{8000}{10\,000}(200) = \boxed{160 \text{ m/s}}$

The distance between the station and the airplane is increasing at a rate of 160 m/s

9. (6 marks)

A farmer wants to fence off a rectangular field of area 3750 square meters. One of the sides borders a dangerous area and requires a double fence (twice the amount of fencing per meter). Find the dimensions of the rectangular field that will minimize the total amount of fencing required.



Let x be the length of the field (in m) along the dangerous area.
Let y be the depth of the field (in m) away from the dangerous area.

The area of the field is $A = xy$, so $xy = 3750$,

so $\boxed{y = \frac{3750}{x}}$

The amount of fencing required is: $F = 3x + 2y = 3x + 2\left(\frac{3750}{x}\right) = 3x + \frac{7500}{x}$

So $\boxed{F(x) = 3x + \frac{7500}{x}}$ is to be minimized.

$F'(x) = 3 - \frac{7500}{x^2} = \frac{3x^2 - 7500}{x^2} = \boxed{\frac{3(x^2-2500)}{x^2}}$

So $F'(x) = 0$ if $x^2 - 2500 = 0$, $x^2 = 2500$, $\underline{x = 50}$ (since $x > 0$)

If $x < 50$, $x^2 < 2500$, so $x^2 - 2500 < 0$, so $\frac{3(x^2-2500)}{x^2} < 0$, so $F'(x) < 0$

If $x > 50$, $x^2 > 2500$, so $x^2 - 2500 > 0$, so $\frac{3(x^2-2500)}{x^2} > 0$, so $F'(x) > 0$

so $F(x)$ is minimized when $\boxed{x = 50}$ and $y = \frac{3750}{50} = \boxed{75}$

The total amount of fencing is minimized when the length of the field is 50 m and its depth is 75 m.

10. (8 marks)

Evaluate the following indefinite integrals

(a) $\displaystyle\int \left(x^{4/3} + 4 \csc x \cot x - \frac{3}{1+x^2} \right) dx$

$$= \frac{x^{7/3}}{\left(\frac{7}{3}\right)} + 4(-\csc x) - 3(\tan^{-1} x) + C$$

$$= \boxed{\frac{3}{7} x^{7/3} - 4\csc x - 3\tan^{-1} x + C}$$

(b) $\displaystyle\int \frac{2e^x}{e^x + 5}\, dx$

Let $u = e^x + 5$, then $du = e^x\, dx$

So $$\int \frac{2e^x}{e^x+5}\, dx = \int \frac{2}{e^x+5}(e^x dx) = \int \frac{2}{u}\, du$$

$$= 2\int \frac{1}{u}\, du = 2\ln|u| + C = 2\ln|e^x+5| + C$$

$$= \boxed{2\ln(e^x+5) + C}$$

11. (8 marks)

Evaluate the following definite integrals

(a) $\displaystyle\int_1^4 \left(3x + 3\sqrt{x} + \frac{4}{x^2}\right)\, dx$

$$\int \left(3x + 3\sqrt{x} + \frac{4}{x^2}\right) dx = \int \left(3x + 3x^{1/2} + 4x^{-2}\right) dx$$

$$= 3\left(\frac{x^2}{2}\right) + 3\left(\frac{x^{3/2}}{\left(\frac{3}{2}\right)}\right) + 4\left(\frac{x^{-1}}{-1}\right) + C$$

$$= \boxed{\frac{3}{2}x^2 + 2x^{3/2} - \frac{4}{x} + C}$$

So $\displaystyle\int_1^4 \left(3x + 3\sqrt{x} + \frac{4}{x^2}\right) dx = \left(\frac{3}{2}x^2 + 2x^{3/2} - \frac{4}{x}\right)\Big|_1^4$

$$= \left(\frac{3}{2}(4)^2 + 2(4)^{3/2} - \frac{4}{4}\right) - \left(\frac{3}{2}(1)^2 + 2(1)^{3/2} - \frac{4}{1}\right) = 24 + 16 - 1 - \frac{3}{2} - 2 + 4$$

$$= 41 - \frac{3}{2} = \boxed{\frac{79}{2}}$$

(b) $\displaystyle\int_1^2 \frac{6x^2 + 4}{\sqrt{x^3 + 2x + 13}}\, dx$

Let $u = x^3 + 2x + 13$, $du = (3x^2 + 2)\,dx$, so

$2\,du = (6x^2 + 4)\,dx$

So $\displaystyle\int \frac{6x^2+4}{\sqrt{x^3+2x+13}}\, dx = \int \frac{1}{\sqrt{u}}(2\,du) = 2\int u^{-1/2}\, du$

$$= 2\,\frac{u^{1/2}}{\left(\frac{1}{2}\right)} + C = 4\sqrt{u} + C = \boxed{4\sqrt{x^3+2x+13} + C}$$

So $\displaystyle\int_1^2 \frac{6x^2+4}{\sqrt{x^3+2x+13}}\, dx = 4\sqrt{x^3+2x+13}\,\Big|_1^2$

$$= 4\sqrt{(2)^3 + 2(2) + 13} - 4\sqrt{(1)^3 + 2(1) + 13} = 4\sqrt{25} - 4\sqrt{16}$$

$$= 4(5) - 4(4) = \boxed{4}$$

12. (5 marks)

A 60-μF capacitor initially has a voltage of 200 V across it. At a certain instant (when t=0), a current $i(t) = 50t$ (where i is measured in mA and t is measured in s) is sent through the circuit containing the capacitor. How long does it take for the voltage across the capacitor to reach 350 V ?

$$V_c = \frac{1}{C}\int i\,dt = \frac{1}{60\times10^{-6}}\int 50t\,(10^{-3})\,dt$$

$$= \frac{50}{60}(1000)\int t\,dt = \frac{2500}{3}\left(\frac{t^2}{2}\right) + C_1 = \frac{1250}{3}t^2 + C_1$$

At $t=0$, $V_c = 200$, so:

$$200 = \frac{1250}{3}(0)^2 + C_1 = C_1, \quad \text{so } \boxed{C_1 = 200}$$

$$\boxed{V_c = \frac{1250}{3}t^2 + 200}$$

So for the voltage across the capacitor to reach 350 V, we have:

$$350 = \frac{1250}{3}t^2 + 200$$

$$\frac{1250}{3}t^2 = 150$$

$$t^2 = 150\left(\frac{3}{1250}\right) = 0.36$$

$$\boxed{t = 0.6 \text{ s}}$$

It takes 0.6 s (or 600 ms) for the voltage to reach 350 V.