DAWSON COLLEGE – DEPARTMENT OF MATHEMATICS
FINAL EXAM, FALL 2019 **CALCULUS II (201-203-DW sections 01, 02, 03)**
DECEMBER 12, 2019 (9:30am-12:30pm)
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INSTRUCTIONS:

The exam has 14 questions on 10 pages.

An information sheet is provided

Make sure "NOW" that all the pages are included with your exam. If that is not the case, notify your teacher immediately

Do not detach any pages from this document. It should be returned intact.

NO MARKS are given for missing or improperly labeled answers

Only Sharp EL-531X – like calculators are allowed

1) Use the Riemann-sum approach to calculate the area under the graph of the function $f(x)=2x^2-x+2$ over the closed interval $[0,3]$. **[5 Marks]**

$$\Delta x = \frac{3}{n} \qquad x_k = \frac{3k}{n}$$

$$f(x_k) = \frac{2 \cdot 9k^2}{n^2} - \frac{3k}{n} + 2$$

$$f(x_k)\Delta x = \frac{54k^2}{n^3} - \frac{9k}{n^2} + \frac{6}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{54}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{9}{n^2}\frac{n(n+1)}{2} + \frac{6n}{n}$$

$$\lim_{n\to\infty}\left\{9\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{9}{2}\left(1+\frac{1}{n}\right) + 6\right\}$$

$$= 18 - \frac{9}{2} + 6 = \frac{48-9}{2} = \frac{39}{2}$$

2) Find y(x) given that $\frac{dy}{dx} = \frac{1+x+x^2}{\sqrt{x}}$ and y(1) = 2 **[5 Marks]**

$$y(x) = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}} + x^{\frac{3}{2}} dx$$

$$= 2\sqrt{x} + \frac{2}{3}\sqrt{x}^3 + \frac{2}{5}\sqrt{x}^5 + C$$

$$2 = 2 + \frac{2}{3} + \frac{2}{5} + C \Rightarrow C = -\frac{2}{3} - \frac{2}{5}$$

$$= -\frac{16}{15}$$

3) The number of subscribers S(t) to an online channel is increasing at the approximate rate of

$$S'(t) = 5000 - \frac{1000}{(t+10)^2}$$

Assuming this projected rate continues what will be the expected number of subscribers 10 months from now (t=10)? **[5 Marks]**

Since the initial number of clients is not given we set it to N, therefore $S(0) = N$

$$S(t) = \int S'(t)dt = 5000t + \frac{1000}{t+10} + C$$

and $N = S(0) = \frac{1000}{10} + C \Rightarrow C = N - 100$

The new number of clients the company will have in 10 month is

$$S(10) - S(0) = 50000 + 50 - 100$$

$$= 49950$$

The company will have then, 49950 + N clients in 10 months.

4) Find the average value of the following function over the $[0,\pi]$ interval (use radians in the calculator) **[6 Marks]**

$$f(x)=\sin(x)\cos^3(x)$$

$$f(x) = \sin(x)\cos^3(x)$$

$$F(x) \doteq \int f(x)\,dx = \int \sin(x)\cos^3(x)\,dx$$

$$= -\int u^3\,du = -\frac{1}{4}\cos^4(x)$$

$$\text{average} = \frac{F(\pi)-F(0)}{\pi-0} = \left\{-\frac{1}{4}\cos^4(\pi)\right.$$

$$\left.+\frac{1}{4}\cos^4(0)\right\}/\pi = 0$$

5) Find the following indefinite integrals **[24 Marks]**

a) $\int \frac{x}{(3-x)^3}dx$ (6 Marks)

$$-\int \frac{3-u}{u^3}du = \int \frac{u}{u^3} - \frac{3}{u^3}\,du = -\frac{1}{u} + \frac{3}{2u^2}$$

$$= -\frac{1}{3-x} + \frac{3}{2}\frac{1}{(3-x)^2} + C$$

b) $\int \frac{2x-1}{(x+2)(x^2+1)}dx$ (6 Marks)

$$A(x^2+1)+(Bx+C)(x+2)=2x-1$$

for $x=-2$

$$5A=-5 \Rightarrow \boxed{A=-1}$$

for $x=0$

$$-1+C2=-1 \Rightarrow \boxed{C=0}$$

for $x=1$

$$-1\cdot 2+B(3)=1$$

$$\Rightarrow B\cdot 3=3$$

$$\Rightarrow \boxed{B=1}$$

So,

$$\int \frac{2x-1}{(x+2)(x^2+1)}dx = \int -\frac{1}{x+2}dx + \int \frac{x}{x^2+1}dx$$

$$= -\ln|x+2| + \frac{1}{2}\ln|x^2+1| + C$$

c) $\int x^3 \ln(x)\, dx$ (6 Marks)

$$= \frac{x^4}{4}\ln(x) - \int \frac{x^4}{4}\cdot\frac{1}{x}\, dx$$

$$= \underline{\frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + C} \quad ✓$$

d) $\int \frac{\cos(\ln x)}{x} dx = \int \cos(u)\, du =$ (6 Marks)

$$\boxed{\begin{aligned} u &= \ln(x) \\ \frac{du}{dx} &= \frac{1}{x} \\ dx &= x\, du \end{aligned}}$$

$$\sin(u) + C = \underline{\sin(\ln x) + C} \quad ✓$$

6) Find the consumer's surplus if the demand function for the product is $D(x)=84-5x^2$ **,** the supply function for the product is $S(x)=x^2-12$ and the current market price is set at the equilibrium price. **[5 Marks]**

$$84-5x^2=x^2-12 \Rightarrow 96=6x^2 \Rightarrow \boxed{\bar{x}=4}$$

$$\bar{p}=\bar{x}^2-12=16-12=4$$

$$\int_0^{\bar{x}=4} 84-5x^2\,dx = 84\cdot 4-\frac{5}{3}4^3=$$

$$\frac{336\cdot 3-5\cdot 64}{3}=\frac{1008-320}{3}$$

$$CS=\frac{1008-320}{3}-\frac{16\cdot 3}{3}=\frac{640}{3}$$ ✓

7) Find the area of the region completely enclosed by the functions $f(x)=x^2-3x+7$ and $g(x)=4x-5$. **[5 Marks]**

$$x^2-3x+7=4x-5 \Rightarrow x^2-7x+12=0$$

$$\Rightarrow (x-3)(x-4)=0 \Rightarrow \boxed{x=3} \text{ and } \boxed{x=4}$$

So,

$$\int_{a=3}^{b=4} -x^2+7x-12\,dx = -\frac{x^3}{3}+\frac{7x^2}{2}-12x\Big|_3^4$$

$$=-\frac{4^3}{3}+\frac{7\cdot 4^2}{2}-12\cdot 4+\frac{3^3}{3}-\frac{7\cdot 3^2}{2}+12\cdot 3$$

$$=-\frac{37}{3}+\frac{49}{2}-12=\frac{-74+147-72}{6}=\underline{\frac{1}{6}}$$ ✓

8) Use the trapezoidal formula to calculate an approximate value of

$\int_1^2 \frac{2}{\sqrt{x+1}}dx$ with $n=4$. Use 3 decimal places in your answer.

[5 marks]

	0	1	2	3	4
x_k	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
$f(x_k)$	$\frac{2}{\sqrt{2}}$	$\frac{2\cdot 2}{3}$	$\frac{2\cdot 2}{\sqrt{10}}$	$\frac{2\cdot 2}{\sqrt{11}}$	$\frac{2}{\sqrt{3}}$

$\Delta x = \frac{1}{4}$

$$\int_1^2 \frac{2}{\sqrt{x+1}}dx \approx \frac{1}{8}\left(\frac{2}{\sqrt{2}} + \frac{2\cdot 2\cdot 2}{3} + \frac{2\cdot 2\cdot 2}{\sqrt{10}} + \frac{2\cdot 2\cdot 2}{\sqrt{11}} + \frac{2}{\sqrt{3}}\right)$$

$$\approx 1.272$$

9) Find the general solution y of the following separable differential equation. Do not isolate.

$$e^{2y}x^3y' = x$$

[5 Marks]

$$\int e^{2y}dy = \int \frac{1}{x^2}dx$$

$$\frac{e^{2y}}{2} = -\frac{1}{x} + C_1 \Rightarrow e^{2y} = -\frac{2}{x} + C$$

10) Calculate the following limit $\lim_{x\to 0}\frac{3x^2-3x+\sin(3x)}{3\cos(2x)+3x-3e^x}$ $\frac{0}{0}$ case **[5 Marks]**

$$= \lim_{x\to 0} \frac{6x-3+3\cos(3x)}{-6\sin(2x)+3-3e^x} \qquad \frac{0}{0}\text{-case}$$

$$= \lim_{x\to 0} \frac{6-9\sin(3x)}{-12\cos(2x)-3e^x} = -\frac{6}{15} = -\frac{2}{5}$$

11) Calculate the following improper integral **[5 Marks]**

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$F(x) = \int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int e^u\,du = -2e^{-\sqrt{x}}$$

So,

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{x\to\infty}\left(-2e^{-\sqrt{x}}\right) + 2e^{-\sqrt{1}}$$

$$= \lim_{x\to\infty}\left(\xrightarrow{0}-\frac{2}{e^{\sqrt{x}}}\right) + \frac{2}{e} = \frac{2}{e}$$

12) Find the **third** Taylor polynomial of $f(x)=\sqrt{2x+4}$ at $x=0$ and use it to find an approximate value for $f(0.5)$. Use 5 decimal places in your answer.
[5 Marks]

$$f'(x)=\frac{1}{2}\,2\,\frac{1}{\sqrt{2x+4}}=\frac{1}{\sqrt{2x+4}} \qquad f'(0)=\frac{1}{2}$$

$$f''(x)=-\frac{1}{2}\,2\,\frac{1}{\sqrt{2x+4}^{\,3}}=-\frac{1}{\sqrt{2x+4}^{\,3}} \qquad f''(0)=-\frac{1}{8}$$

$$f'''(x)=\frac{3}{2}\,2\,\frac{1}{\sqrt{2x+4}^{\,5}}=\frac{3}{\sqrt{2x+4}^{\,5}} \qquad f'''(0)=\frac{3}{32}$$

$$P_3(x)=2+\frac{1}{2}x-\frac{1}{8\cdot 2}x^2+\frac{3}{32\cdot 6}x^3$$

$$=\boxed{2+\frac{1}{2}x-\frac{1}{16}x^2+\frac{1}{64}x^3} \qquad \boxed{P_3(0.5)\approx 2.23633}$$

13) Does the series $\sum_{n=0}^{\infty}\frac{5\cdot 3^{n+2}-1}{6^{n-1}}$ converge or diverge? If it converges find its value. **[5 Marks]**

$$5\cdot 9\cdot 6\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-6\sum_{n=0}^{\infty}\left(\frac{1}{6}\right)^n$$

$$=\frac{270}{1-\frac{1}{2}}-\frac{6}{1-\frac{1}{6}}=270\cdot 2-\frac{6\cdot 6}{5}$$

$$=\frac{540\cdot 5-36}{5}=\frac{2700-36}{5}=\frac{2664}{5}$$

14) Determine whether the following series converges or diverges. State each test used and explain each solution based on it **[15 Marks]**

a) $\sum_{n=1}^{\infty} \frac{n^3-2n+1}{5-n+4n^3}$ Diverges (5 Marks)

Divergence Test

$$\lim_{n\to\infty} \frac{n^3-2n+1}{5-n+4n^3} = \frac{1}{4} \neq 0$$

b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^5}-10}$ Diverges (5 Marks)

Comparison Test

$$\underbrace{\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^5}}}_{p\text{-series}} < \sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^5-10}}$$

p-series $p = \frac{2}{3} < 1$ (Diverges)

c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ Diverges (5 Marks)

Integral Test

$$F(x) = \int \frac{\ln(x)}{x} dx = \int u \, du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$\lim_{x\to\infty} F(x) = \infty \text{ (Diverges)}$$