Dawson College
Mathematics Department

Final Examination

Linear Algebra
201-105-DW

Sections 01, 02, 03, 04, 05

May 23, 2012

9:30 am-12:30 pm

Student Name: ____________________________________

Student ID: ____________________________________

Teachers: G. Honnouvo, A. Jimenez, V. Ohanyan

TIME: 3 Hours

Instructions:

- Print your name and student ID number in the space provided above
- All questions are to be answered directly on the examination paper in the space provided
- Translation and regular dictionaries are permitted
- Small, noiseless and non-programmable calculators are permitted

This examination consists of 15 pages.
Please ensure that you have a complete examination before starting.

This exam must be returned intact.

1a) (5 marks) Solve the following linear system.

$$\begin{cases} x_1 + 2x_2 + 8x_3 + 4x_4 = 2 \\ 2x_1 + 5x_2 + 19x_3 + 8x_4 = 4 \\ 3x_1 + 3x_2 + 16x_3 + 6x_4 = 8 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 8 & 4 & 2 \\ 2 & 5 & 19 & 8 & 4 \\ 3 & 3 & 16 & 6 & 8 \end{array}\right) \begin{array}{l} \\ -2R_1 \\ -3R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 8 & 4 & 2 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & -3 & -8 & -6 & 2 \end{array}\right) \begin{array}{l} \\ \\ 3R_2 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & 8 & 4 & 2 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -6 & 2 \end{array}\right)$$

$$\begin{cases} \boxed{x_4 = \text{parameter}} \\ \boxed{x_3 = 2 + 6x_4} \\ \boxed{x_2} = -3x_3 = -3(2+6x_4) = \boxed{-6 - 18x_4} \\ x_1 = 2 - 4x_4 - 8x_3 - 2x_2 \end{cases}$$

$$= 2 - 4x_4 - 16 - 48x_4 + 12 + 36x_4$$

$$\boxed{x_1 = -2 - 16x_4}$$

1b) (2 marks) Find a particular solution where $x_2 = 12$

$$12 = -6 - 18x_4 \Rightarrow \boxed{x_4 = -1}$$

$$\begin{cases} x_4 = -1 \\ x_3 = -4 \\ x_2 = 12 \\ x_1 = 14 \end{cases}$$

2. (4 marks) For which values of k the system will have

$$\begin{cases} x+y+2z=k \\ 3x+y+7z=3k+4 \\ 4x+2y+(k+7)z=5k \end{cases}$$

a) Exactly one solution, b) No solution, c) Infinitely many solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & k \\ 3 & 1 & 7 & 3k+4 \\ 4 & 2 & k+7 & 5k \end{array}\right) \begin{array}{l} \\ -3R_1 \\ -4R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & k \\ 0 & -2 & 1 & 4 \\ 0 & -2 & k-1 & k \end{array}\right) \begin{array}{l} \\ \\ -R_2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 2 & k \\ 0 & -2 & 1 & 4 \\ 0 & 0 & k-2 & k-4 \end{array}\right)$$

a) $k-2 \neq 0 \Rightarrow \boxed{k \neq 2}$

b) $k-2=0$ and $k-4 \neq 0$ $\boxed{k=2}$

c) $k-2=0$ and $k-4=0$ Not possible

3. (3 marks) Let A and B be $(n \times n)$ matrices such that $B = I - A$ and $A^2 = A$. Show that $AB = BA = 0$.

$$AB = A(I-A) = A - A^2 = A - A = 0$$

$$BA = (I-A)A = A - A^2 = A - A = 0$$

4. (3+3+4 marks) Given $A=\begin{pmatrix}1&3\\2&5\end{pmatrix}$, $B=\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix}$, $C=\begin{pmatrix}1&2&3\\0&1&-1\end{pmatrix}$.

a) Find $tr(ACB-2I_2)=5$

$$\boxed{\begin{aligned} AC &= \begin{pmatrix}1&5&0\\2&9&1\end{pmatrix} \\ ACB &= \begin{pmatrix}6&1\\11&3\end{pmatrix} \\ ACB-2I &= \begin{pmatrix}4&1\\11&1\end{pmatrix}\end{aligned}}$$

b) Find $\det(CB)$

$$CB = \begin{pmatrix}3&4\\1&-1\end{pmatrix}$$

$$= \frac{1}{-3-4}\begin{pmatrix}-1&-4\\-1&3\end{pmatrix}$$

c) Find the matrix X such that $(X^T+2I_2)^{-1}=A$

$$X^T+2I_2 = A^{-1} = \frac{1}{-1}\begin{pmatrix}5&-3\\-2&1\end{pmatrix} = \begin{pmatrix}-5&3\\2&-1\end{pmatrix}$$

$$X^T = \begin{pmatrix}-7&3\\2&-3\end{pmatrix}$$

$$\boxed{X = \begin{pmatrix}-7&2\\3&-3\end{pmatrix}}$$

5. (4+3 marks) Given $A=\begin{pmatrix}1&0&1\\2&1&0\\2&1&-1\end{pmatrix}$, $b=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

a) Find $adj(A)$ $=\begin{pmatrix}-1&2&0\\1&-3&-1\\-1&2&1\end{pmatrix}^T=\begin{pmatrix}-1&1&-1\\2&-3&2\\0&-1&1\end{pmatrix}$

b) Use $adj(A)$ to solve the system $AX=b$.

$$|A|=\begin{vmatrix}1&0&1\\2&1&0\\3&1&0\end{vmatrix}=\begin{vmatrix}2&1\\3&1\end{vmatrix}=-1$$

$$A^{-1}=\frac{1}{-1}\,adj(A)=\begin{pmatrix}1&-1&1\\-2&3&-2\\0&1&-1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=A^{-1}\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1&-1&1\\-2&3&-2\\0&1&-1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix}$$

6. (4 marks) Evaluate the determinant.

$$\begin{vmatrix} 1 & 0 & -1 & 5 \\ -4 & 4 & -6 & 1 \\ 2 & -2 & 3 & 0 \\ -2 & 0 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 \\ 2 & -2 & 3 & 0 \\ -2 & 0 & 4 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 2 & -2 & 3 \\ -2 & 0 & 4 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} = -2 \cdot 2 = -4$$

7. Let A and B be (2×2) matrices, where $\det(A)=2$, $\det(B)=3$. Find

a) (2 marks) $\det\left((2A)^{-1}B^2\right)$ $= |(2A)^{-1}||B^2| = \frac{1}{|2A|}|B||B|$

$= \frac{3^2}{2^2|A|} = \frac{3^2}{2^3}$

b) (2 marks) $\det\left(2A^{-1}+7adj(A)\right)$ $= |2A^{-1}+7A^{-1}|A||$

$= |(2+7\cdot 2)A^{-1}| = 16^2 \frac{1}{|A|} = \frac{16^2}{2}$

c) (3 marks) $\det\left(B^T A^{-3}(2BA)^2\right)$ $= |B^T||A^{-3}||(2BA)^2|$

$= |B| \frac{1}{|A^3|} |2BA||2BA| = \frac{3\cdot 2^2|B||A|2^2|B||A|}{|A|^3}$

$= \frac{3\cdot 2^2\cdot 2^4 3^2\cdot 2^2}{2^3} = 3^3 2^5$

8. (4+4 marks) Let $\vec{u}=(1, \quad 2, \quad 3)$, $\vec{v}=(-1, \quad 2, \quad k)$, $\vec{w}=(3, \quad 1, \quad 2)$.

a) For which values of k the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular

$$\vec{u}+\vec{v} = (0, 4, 3+k)$$

$$\vec{u}-\vec{v} = (2, 0, 3-k)$$

$$(\vec{u}+\vec{v})(\vec{u}-\vec{v}) = (3+k)(3-k) = 0$$

$$\boxed{k = \pm 3}$$

b) Find $Proj_{2\vec{w}}(\vec{u}+\vec{w})$

$$\vec{u}+\vec{w} = (4, 3, 5) \qquad 2\vec{w} = (6, 2, 4)$$

$$proj_{2\vec{w}}(\vec{u}+\vec{w}) = \frac{(\vec{u}+\vec{w})(2\vec{w})}{\|2\vec{w}\|^2} 2\vec{w}$$

$$= \frac{4\cdot 6 + 3\cdot 2 + 5\cdot 4}{6^2+2^2+4^2}(6, 2, 4)$$

$$\frac{50}{56}(6, 2, 4)$$

9. (3 marks) Let $\vec{u}=(-1, \quad 2, \quad 3)$, $\vec{v}=(-1, \quad 2, \quad 1)$. Find a unit vector perpendicular to the vectors $\vec{u}+2\vec{v}$ and $\vec{v}$

$2\vec{v}=(-2,4,2)$ $\quad \vec{u}+2\vec{v}=(-3,6,5)$

$\vec{w}=\vec{v}\times(\vec{u}+2\vec{v}) = (-1,2,1)\times(-3,6,5)$

$=(4,2,0)$

unit vector in the direction of $\vec{w}$

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{20}}(4,2,0)$$

10. (4 marks) Find the area of the parallelogram determined by the vectors $\vec{u}=(1, \quad -1, \quad 0)$, and $\vec{v}=(0, \quad 1, \quad -1)$

$A_{\square} = \|\vec{u}\times\vec{v}\| = \|(1,1,1)\| = \sqrt{3}$

11a) (3 marks)) Find the line of intersection of the planes $x - y + z = -1$ and $2x - y + 3z = 1$.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 2 & -1 & 3 & 1 \end{array}\right) - 2R_1 \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

$$l: \begin{cases} z = t \quad \text{(parameter)} \\ y = 3 - t \\ x = -1 - t + y = -1 - t + 3 - t = 2 - 2t \end{cases}$$

12b) (3 marks) Find the parametric equations of the line passing through the point $A(2, 1, 1)$ and parallel to the line of intersection of the planes $x - y + z = -1$ and $2x - y + 3z = 1$ in part a)

$\vec{v} = (-2, -1, 1)$

$A(2, 1, 1)$

$l: (2 - 2t, 1 - t, 1 + t)$

12. (4 marks) Find the equation of the plane containing the lines $\begin{cases} x = 1+t \\ y = t \\ z = 1-t \end{cases}$ and $\begin{cases} x = 2-s \\ y = 1+s \\ z = s \end{cases}$

$\hat{n} = (1,1,-1) \times (-1,1,1) = (2,0,2)$

$P_0 = (1,0,1) \quad (t=0)$

$p: 2(x-1) + 2(z-1) = 0$ or $2x+2z-4=0$ ...

13. (4 marks) Find the intersection of the plane $2x + y + 2z = 11$ and the line $\begin{cases} x = 1-2t \\ y = 2-t \\ z = 3+3t \end{cases}$

where t is a real number.

$2(1-2t) + (2-t) + 2(3+3t) = 11$

$-4t - t + 6t + 2 + 2 + 6 = 11$

$t = 1$

Intersection point $P(-1, 1, 6)$

14. (3+3 marks) Given $\vec{u}\cdot(\vec{v}\times\vec{w})=5$. Find

a) $\vec{w}\cdot(2\vec{v}\times 3\vec{u}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ 2v_1 & 2v_2 & 2v_3 \\ 3u_1 & 3u_2 & 3u_3 \end{vmatrix} = -6 \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$= -6\cdot 5 = -30$

b) $(\vec{v}\times\vec{u})\cdot(\vec{u}-\vec{v}) = 0$

15. (4 marks) Find the equation of the plane containing the points $A(2,\ 0,\ -1)$, $B(0,\ 1,\ 2)$ and perpendicular to the plane $2x+y+z=1$.

$\vec{n} = \vec{AB} \times (2, 1, 1) = (-2, 1, 3) \times (2, 1, 1) = (-2, 8, -4)$

$P_0 = A(2, 0, -1)$

$p:\ -2(x-2) + 8y - 4(z+1) = 0$ or

$-2x + 8y - 4z = 0$

16. (4 marks) Find the distance between the point $A(2, \quad 1, \quad 1)$ and the line $\begin{cases} x = 1 + t \\ y = 2 - 3t \\ z = 1 + 2t \end{cases}$

$B(1, 2, 1) \quad (t=0)$

$C(2, -1, 3) \quad (t=1)$

$$d = \frac{\| \vec{BA} \times \vec{BC} \|}{\| \vec{BC} \|}$$

$\vec{BA} = (1, -1, 0)$

$\vec{BC} = (1, -3, 2)$

$\vec{BA} \times \vec{BC} = (-2, -2, -2)$

$$d = \frac{\sqrt{3 \cdot 2^2}}{\sqrt{14}} = \frac{2\sqrt{3}}{\sqrt{14}} = 2\sqrt{\frac{3}{14}} \quad \ldots$$

17. (8 marks) Maximize the profit function $P = 15x_1 + 50x_2 - 45x_3$ subject to

$$\begin{cases} x_1 + 2x_2 - x_3 \le 3 \\ x_1 + x_2 - x_3 \le 1 \\ 2x_1 - 2x_2 - x_3 \le 2 \end{cases} \qquad \textit{where } x_1 \ge 0,\ x_2 \ge 0, x_3 \ge 0$$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & \boxed{1} & -1 & 1 \\ 2 & -2 & -1 & 2 \\ \hline -15 & -50 & 45 & 0 \\ & \uparrow & & \end{array} \qquad \begin{array}{c} \underline{\text{Ratios}} \\ 3/2 \\ 1 \\ \times \\ \\ \\ \end{array}$$

$$\begin{array}{ccc|c} -1 & 0 & \boxed{1} & 1 \\ 1 & 1 & -1 & 1 \\ 4 & 0 & -3 & 4 \\ \hline 35 & 0 & -5 & 50 \\ & & \uparrow & \end{array} \qquad \begin{array}{c} \text{Ratios} \\ 1 \\ \times \\ \times \\ \\ \\ \end{array}$$

$$\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 7 \\ \hline 30 & 0 & 0 & 55 \end{array}$$

$x_1 = 0$

$x_2 = 2$

$x_3 = 1$

$P = 15 \cdot 0 + 50 \cdot 2 - 45 \cdot 1$

$= 100 - 45 = 55$

18. (7 marks) Minimize the cost function $C = 12x_1 + 40x_2 + 30x_3$ subject to

$$\begin{cases} x_1 + 2x_2 + 2x_3 \geq 2 \\ -x_1 - x_2 - 3x_3 \geq -1 \\ -x_1 + 2x_2 + x_3 \geq -2 \end{cases} \qquad \textit{where } x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0$$

$$\begin{array}{ccc|ccc|c} \boxed{1} & -1 & -1 & 1 & 0 & 0 & 12 \\ 2 & -1 & 2 & 0 & 1 & 0 & 40 \\ 2 & -3 & 1 & 0 & 0 & 1 & 30 \\ \hline -2 & 1 & 2 & 0 & 0 & 0 & \end{array}$$

$$\begin{array}{ccc|ccc|c} 1 & -1 & -1 & 1 & 0 & 0 & 12 \\ 0 & \boxed{1} & 4 & -2 & 1 & 0 & 16 \\ 0 & -1 & 3 & -2 & 0 & 1 & 6 \\ \hline 0 & -1 & 0 & 2 & 0 & 0 & 24 \end{array}$$

$$\begin{array}{ccc|ccc|c} 1 & 0 & 3 & -1 & 1 & 0 & 28 \\ 0 & 1 & 4 & -2 & 1 & 0 & 16 \\ 0 & 0 & 7 & -4 & 1 & 1 & 22 \\ \hline 0 & 0 & 4 & 0 & 1 & 0 & 40 \\ & & & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ & & & x_1 & x_2 & x_3 & \text{Min} \end{array}$$