DAWSON
COLLEGE

MATHEMATICS DEPARTMENT
Calculus I (Social Science)

201-103-DW
Winter 2019
Final Examination
May 17th, 2019
Time Limit: 3 hours

Name: Solution

ID#:

Instructors: I. Gorelyshev, M. Hitier, M. Moodi, N. Sabetghadam

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- ▷ This exam contains 18 pages (including this cover page) and 13 problems. Check to see if any pages are missing.
- ▷ Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- ▷ Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- ▷ You are only permitted to use the **Sharp EL-531XT**, **Sharp EL-531XG** or **Sharp EL-531X** calculator.
- ▷ This examination booklet must be returned intact.
- ▷ Good luck!

Question	Points	Score
1	12	
2	7	
3	6	
4	4	
5	16	
6	6	
7	4	
8	8	
9	7	
10	4	
11	6	
12	14	
13	6	
Total:	100	

[12 marks] 1. Find each of the following limits

(a) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 + x - 2} = \dfrac{0}{10} = \boxed{0}$

(b) $\lim\limits_{x \to \infty} \dfrac{x^2 - x^4}{5x^4 - 2x + 1} = \lim\limits_{x \to \infty} \dfrac{\cancel{x^4}\left(\frac{1}{x^2} - 1\right)}{\cancel{x^4}\left(5 - \frac{2}{x^3} + \frac{1}{x^4}\right)} = \dfrac{0 - 1}{5 - 0 + 0} = \boxed{-\dfrac{1}{5}}$

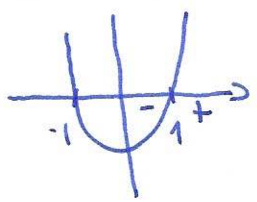
(c) $\displaystyle\lim_{x\to 5} \frac{2x-10}{\sqrt{x-1}-2} \underset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 5} \frac{2(x-5)(\sqrt{x-1}+2)}{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}$

$\displaystyle = \lim_{x\to 5} \frac{2\cancel{(x-5)}(\sqrt{x-1}+2)}{\cancel{(x-1-4)}}$

$= 2(\sqrt{5-1}+2) = \boxed{8}$

(d) $\displaystyle\lim_{x\to 1^-} \frac{x-2}{x^2-1} = \text{"}\frac{-1}{0^-}\text{"} = \boxed{\infty}$

$x^2 - 1 = (x-1)(x+1)$



[7 marks] 2. Consider the piecewise defined function $f(x) = \begin{cases} 2\dfrac{x-2}{x^2-4} & x < 2 \\ \frac{1}{4}\sqrt{5x-6} & x \geq 2 \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2\frac{x-2}{x^2-4} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 2^-} 2\frac{\cancel{x-2}}{(\cancel{x-2})(x+2)}$$

$$= \frac{2}{2+2} = \boxed{\frac{1}{2}}$$

(b) Show that f continuous at $x = 2$. (Show all your work)

$$\begin{cases} \lim_{x\to 2^-} f(x) = \frac{1}{2} \\ f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{4}\sqrt{5x-6} = \lim_{x\to 2^+} \frac{1}{4}\cdot\sqrt{4} = \frac{1}{2} \end{cases}$$

$\Rightarrow \lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x) \Rightarrow$ f is continuous at $x = 2$

(c) Where is f discontinuous? Justify briefly.

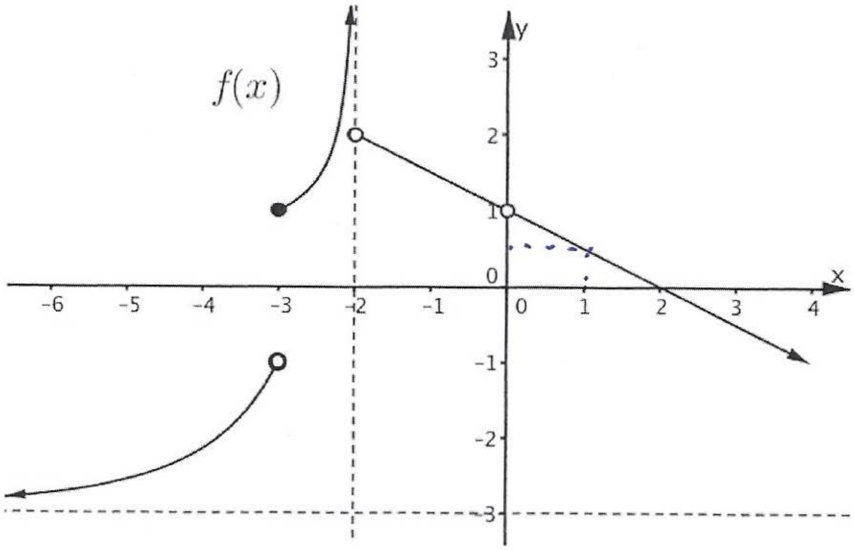
$x^2 - 4 = 0 \Leftrightarrow x = \pm 2$

$x = -2 < 2 \Rightarrow f$ is not defined at $x = -2$

$\Rightarrow$ discontinuous at $x = -2$

(f is continuous on $(-\infty,-2), (-2,2), [2,\infty)$ (defined algebraic))

[6 marks] 3.



The graph of a function f is given above, **state** the following:

Use ∞ or $-\infty$ if necessary and write DNE if it does not exist.

(a) i. $\lim_{x\to -3^-} f(x) = -1$

ii. $f(-3) = 1$

iii. $\lim_{x\to 2} f(x) = 0$

iv. $\lim_{x\to -2^+} f(x) = 2$

v. $f(0)$ DNE

vi. $\lim_{x\to 0} f(x) = 1$

vii. $\lim_{x\to \infty} f(x) = -\infty$

viii. $\lim_{x\to 1} f(x) = \frac{1}{2}$

(b) i. The equation of horizontal asymptote(s) (if any)

$y = -3$

ii. The equation of vertical asymptote(s) (if any)

$x = -2$

(c) The value(s) of x where the function is discontinuous.

$x = -3, -2, 0$

[4 marks] 4. Consider the function $f(x) = 3x^2 - 2x + 1$.

Find the $f'(x)$ using ONLY the limit definition (4-step process) of the derivative.

No marks are given for using the differentiation rules.

$f(x) = 3x^2 - 2x + 1$

1 $f(x+h) = 3(x+h)^2 - 2(x+h) + 1$

$= 3(x^2 + 2xh + h^2) - 2x - 2h + 1$

$= 3x^2 + 6xh + 3h^2 - 2x - 2h + 1$

2 $f(x+h) - f(x) = (\cancel{3x^2} + 6xh + 3h^2 - \cancel{2x} - 2h + \cancel{1}) - (\cancel{3x^2} - \cancel{2x} + \cancel{1})$

$= 6xh + 3h^2 - 2h = (6x + 3h - 2)h$

3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(6x + 3h - 2)\cancel{h}}{\cancel{h}}$

4 $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h - 2)$

$\boxed{f'(x) = 6x - 2}$

[16 marks] 5. Find the derivative of the following functions (do not simplfy)

(a) $f(x) = (x-1)^3(x^2+5)^6$

$f'(x) = 3(x-1)^2(x^2+5)^6 + (x-1)^3 \cdot 6(x^2-5)^5 \cdot 2x$

(b) $g(x) = \dfrac{2-3x^2}{x^3+12}$

$g'(x) = \dfrac{(-6x)(x^3+12)-(2-3x^2)(3x^2)}{(x^3+12)^2}$

(c) $T(x) = \tan^2(x) - \arctan(x^2)$ $= \left(\tan(x)\right)^2 - \arctan(x^2)$

$$T'(x) = 2\tan(x)\cdot\sec^2(x) - \frac{1}{1+(x^2)^2}\cdot 2x$$

(d) $y = \sqrt{2} + x^e + e^x + \ln\left(\sqrt{x} - 2\right)$

$$y' = 0 + e\,x^{e-1} + e^x + \frac{1}{\sqrt{x}-2}\cdot\frac{1}{2\sqrt{x}}$$

[6 marks] 6. Consider the function $$f(x) = \ln\left(\frac{\sqrt{x+1}}{(x+2)\cos^2 x}\right)$$

(a) Fully simplify $f(x)$ using the properties of logarithms.

$$f(x) = \frac{1}{2}\ln(x+1) - \ln(x+2) - 2\ln(\cos x)$$

(b) Compute the value of $f'(x)$ when $x = 0$.

$$f'(x) = \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{x+2} - \frac{2(-\sin x)}{\cos x}$$

$$f'(0) = \frac{1}{2}\cdot 1 - \frac{1}{2} + 2\frac{\sin 0}{\cos 0}$$

$$\boxed{f'(0) = 0}$$

[4 marks] 7. Find the second derivative $f''(x)$ of the function $f(x) = \left(x^4 + x^2 - 5\right)^{3/2}$ and simplify.

$$f'(x) = \frac{3}{2}\left(x^4 + x^2 - 5\right)^{1/2} \cdot \left(4x^3 + 2x\right)$$

$$f''(x) = \frac{3}{2} \cdot \frac{1}{2}\left(x^4 + x^2 - 5\right)^{-1/2}\left(4x^3 + 2x\right) \cdot \left(4x^3 + 2x\right)$$

$$+ \frac{3}{2}\left(x^4 + x^2 - 5\right)^{1/2} \cdot \left(12x^2 + 2\right)$$

$$= \frac{3}{4}\,\frac{\left(4x^3 + 2x\right)^2}{\left(x^4 + x^2 - 5\right)^{1/2}} + \frac{3 \cdot 2}{2 \cdot 2}\,\frac{\left(x^4 + x^2 - 5\right)^{\cancel{1/2}} 2\left(6x^2 + 1\right)}{\left(x^4 + x^2 - 5\right)^{1/2}}$$

$$= \frac{3}{4}\,\frac{16x^6 + 16x^4 + 4x^2 + 4\left(6x^6 + x^4 + 6x^4 + x^2 - 30x^2 - 5\right)}{\left(x^4 + x^2 - 5\right)^{1/2}}$$

$$= \frac{3 \cdot \cancel{4}}{\cancel{4}}\,\frac{4x^6 + 4x^4 + x^2 + 6x^6 + 7x^4 - 29x^2 - 5}{\left(x^4 + x^2 - 5\right)^{1/2}}$$

$$= 3\,\frac{10x^6 + 11x^4 - 28x^2 - 5}{\left(x^4 + x^2 - 5\right)^{1/2}}$$

[8 marks] 8. William is selling cakes at a fixed price of $p = 5.00$ \$
He estimates the cost of producing x cakes a day to be $C(x) = 300 - 0.4x + 0.01x^2$.

(a) Find the revenue and the profit function.

$R(x) = 5 \cdot x$ (revenue)

$P(x) = R(x) - C(x) = 5x - (300 - 0.4x + 0.01x^2)$

$= -300 + 5.4x - 0.01x^2$

(b) Find the marginal profit function.
Use it to estimate the profit from selling the 105^{th} cake.

$P'(x) = 5.4 - 0.02x$

$P'(104) = 5.4 - 0.02 \cdot 104 = 5.4 - 2.08 = 3.32$ \$

The profit from selling the 105^{th} cake is approximately 3.32 \$

(c) Knowing that William won't produce more than 300 cakes a day, find the number of cakes he should bake to maximize his profit.

Domain: $[0, 300]$

$P'(x) = 5.4 - 0.02x = 0 \Leftrightarrow 5.4 = 0.02x$

$\Leftrightarrow x = \frac{5.4}{0.02} = 270$

P' is defined on $[0, 300]$

$P(0) = -300$

$P(270) = 429 \leftarrow$ Absolute maximum.

$P(300) = 420$

William should bake 270 cakes.

[7 marks] 9. Consider the relation $2xy^2 - y^3 = x^2 + 5$.

(a) Find the derivative $\dfrac{dy}{dx}$.

$$\frac{d}{dx}(2xy^2 - y^3) = \frac{d}{dx}(x^2+5)$$

$$2y^2 + 2x\cdot 2yy' - 3y^2y' = 2x$$

$$(4xy - 3y^2)y' = 2x - 2y^2$$

$$\boxed{y' = \frac{2x-2y^2}{4xy-3y^2}}$$

(b) Find the equation of the tangent line to the graph of this relation at the point $(2, 3)$

When $x = 2$ and $y = 3$:

$$y' = \frac{2(2) - 2\cdot 3^2}{4(2)(3) - 3(3)^2} = \frac{4-18}{24-27} = \frac{-14}{-3} = \frac{14}{3}$$

$$y - 3 = \frac{14}{3}(x-2)$$

$$y = \frac{14}{3}x - \frac{28}{3} + \frac{9}{3}$$

$$\boxed{y = \frac{14}{3}x - \frac{19}{3}}$$

[4 marks] 10. Use logarithmic differentiation to find the derivative to the function $y = (\cos x)^{\sin x}$

$$\ln y = \ln(\cos x^{\sin x})$$

$$= \sin x \cdot \ln(\cos x)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}(\sin x \cdot \ln(\cos x))$$

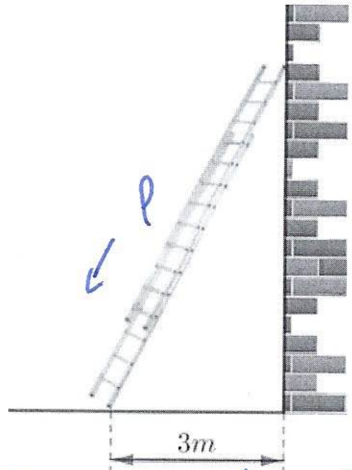
$$\frac{y'}{y} = \cos x \ln(\cos x) + \sin x \cdot -\frac{\sin x}{\cos x}$$

$$y' = \left(\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right)(\cos x)^{\sin x}$$

[6 marks] 11. A telescopic ladder is resting against a wall, its feet resting 3m away from the foot of the wall.

Being not properly secured, the ladder starts to collapse, so that its foot remains 3m away from the wall, the top of the ladder sliding down the wall.

If the ladder collapses (i.e. decreases in length) at a rate of 1.5m/s, how fast is the top of the ladder approaching the ground when the ladder is 5m long?



Want: $\frac{dy}{dt}$ Knowing: $\frac{dl}{dt} = -1.5\ m/s$ when: $l = 5m$

How: $l^2 = 3^2 + y^2 \Rightarrow l^2 = 9 + y^2$

$2l\frac{dl}{dt} = 2y\frac{dy}{dt}$

$5 \cdot (-1.5) = 4\frac{dy}{dt}$

$\frac{dy}{dt} = -\frac{7.5}{4} = -1.875\ m/s$

$5^2 = 9 + y^2$

$25 - 9 = y^2$

$y = 4 \quad (y > 0)$

The top of the ladder is approaching the ground at 1.875 m/s

[14 marks] 12. Given the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2}$

and its derivatives $f'(x) = \dfrac{3x+8}{x^3}$ and $f''(x) = -6\dfrac{x+4}{x^4}$

(a) Find the intercepts (if any)

x-intercept(s): $f(x) = 0 \iff x^2 - 3x - 4 = 0$

$\iff (x-4)(x+1) = 0$

$\iff x = 4$ or $x = -1$ $\boxed{(4,0) \ \& \ (-1,0)}$

y-intercept: $f(0)$ DNE $\Rightarrow$ $\boxed{\text{No } y\text{-intercept}}$

(b) Find the horizontal and vertical asymptote(s) (if any)

H.A.: $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \left(1 - \frac{3}{x} - \frac{4}{x^2}\right) = 1 \Rightarrow \boxed{y = 1}$

V.A: $f(x)$ DNE $\iff x = 0$

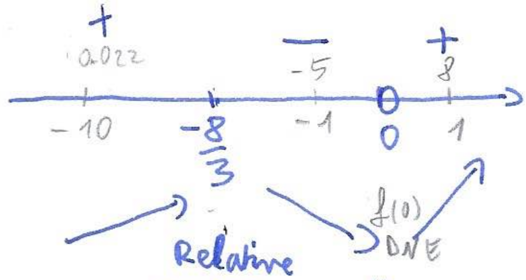
$\lim\limits_{x \to 0} f(x) = \text{"}\frac{-4}{0^+}\text{"} = -\infty \Rightarrow \boxed{x = 0}$

$$f(x) = \frac{x^2 - 3x - 4}{x^2} \qquad f'(x) = \frac{3x+8}{x^3} \qquad f''(x) = -6\frac{x+4}{x^4}$$

(c) Find the intervals where the function is increasing/decreasing and the relative extrema (if any).

$f'(x)$ DNE $\Leftrightarrow x = 0$

$f'(x) = 0 \Leftrightarrow 3x + 8 = 0 \Leftrightarrow x = -\frac{8}{3}$



Relative max

$\left(-\frac{8}{3}, \frac{25}{16}\right)$

$= \left(-\frac{8}{3}, 1.5625\right)$

No relative minimum

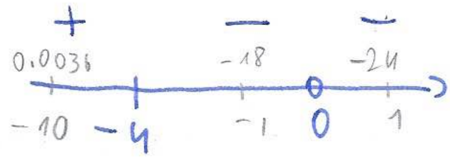
f is increasing on $(-\infty, -\frac{8}{3})$ and $(0, \infty)$

f is decreasing on $(-\frac{8}{3}, 0)$

(d) Find the intervals where the function is concave upward/downward and the inflection points (if any).

$f''(x)$ DNE $\Leftrightarrow x = 0$

$f''(x) = 0 \Leftrightarrow x + 4 = 0 \Leftrightarrow x = -4$



Inflection point:

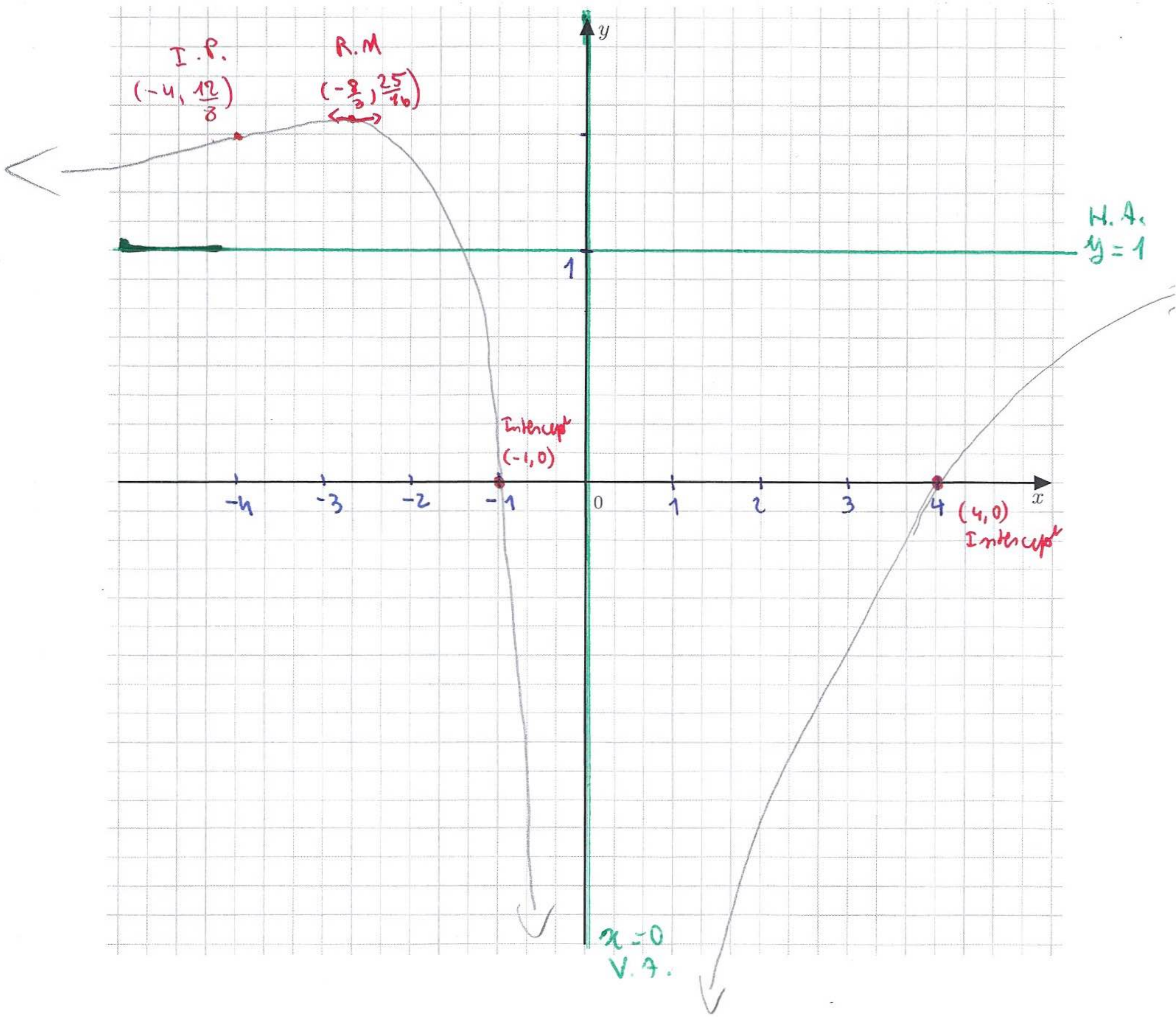
$\left(-4, \frac{24}{16}\right) = (-4, 1.5)$

(change in concavity)

f is concave up on $(-\infty, -4)$

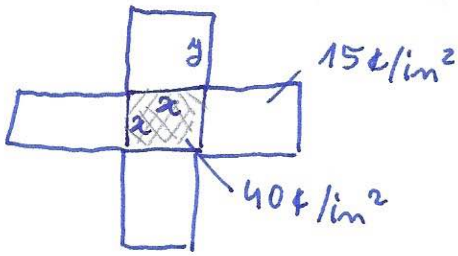
f is concave down on $(-4, 0)$ and $(0, \infty)$

(e) Sketch the function's graph, showing the information found in the previous parts.



x	$-\infty$	-4	$-\frac{8}{3}$	-1	0	4	∞
f'		$+$		$-$		$+$	
f''	$+ \cup +$		$- \frown -$			$- \frown -$	
f	$1 \nearrow$	$\frac{24}{16}$	$\nearrow \frac{25}{16}$	$\searrow 0 \searrow -\infty$	$\Vert$	$-\infty \nearrow 0 \nearrow 1$	
		I.P. (Inflection point)	R.M (relative max)				

[6 marks] 13. A pencil cup with a capacity of 36 in^3 is to be constructed in the shape of a rectangular box with a square base and an open top. If the material for the sides costs 15¢/in^2 and the material for the base costs 40¢/in^2, what should the dimensions of the cup be to minimize the construction cost?



Minimize: construction cost

$C = 40x^2 + 4 \cdot xy \cdot 15$

$C = 40x^2 + 60xy$

Constraint: $x^2 y = 36 \text{ in}^3$

$y = \frac{36}{x^2}$

Domain: $x > 0$, $y > 0$

$C(x) = 40x^2 + 60x \cdot \frac{36}{x^2} = 40x^2 + \frac{2160}{x} = 40\left(x^2 + \frac{54}{x}\right)$

Differentiable on $(0, \infty)$

$C'(x) = 40\left(2x - \frac{54}{x^2}\right) = 80\left(x - \frac{27}{x^2}\right)$

$C'(x) = 0 \Rightarrow x - \frac{27}{x^2} = 0 \Rightarrow x^3 = 27 \Rightarrow \underline{x = 3}$

$C''(x) = 80\left(1 + \frac{54}{x^3}\right)$ $\quad C''(3) = 80(1 + 2) > 0$

By the 2^{nd} derivative Test, C has a relative minimum at $x = 3$ $\left(y = \frac{36}{3^2} = 4\right)$

$\Rightarrow$ The dimension of the cup with minimal construction cost would be $\underbrace{3 \times 3}_{\text{base}} \times 4$ inches